Multi-Regime Shallow Free-Surface Flow Models for Quasi-Newtonian Fluids

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Abstract

The mathematical modeling of thin free-surface laminar flows for quasi-Newtonian fluids (power-law rheology) is addressed with a particular attention to geophysical flows (e.g. ice or lava flows). Asymptotic thin-layer flow models (one-equation and two-equation models) consistent with various viscous regimes, corresponding to different basal boundary conditions (from adherence to pure slip), are derived. The challenge being to derive models consistent from slip to no-slip basal boundary condition, though at the price of balancing small friction by small mean slope. Starting from reference flows (the steady-state uniform ones) corresponding to different shear regimes, the exact expressions of all fields $(\sigma, \mathbf{u}, \mathbf{p})$ are calculated formally by a perturbation expansion method. The calculations are such that all field expressions remain valid for any laminar viscous regimes. The calculations are presented either in a mean slope coordinate system with local variations of the topography or in the Prandtl coordinate system, hence valid in presence of any non flat basal topography. Formal error estimates proving the consistency of the derivations are stated. An unified one-equation model (lubrication type in the depth variable h) is derived at order 1. Next, few unified two-equation models in variable (q, h) (shallow water type) are stated and discussed. The classical first order models from the literature are recovered if considering the corresponding particular cases (generally, flat bottom with a particular regime and/or specific basal boundary condition). Two one-dimensional numerical examples illustrate the robustness of these new multi-regime formulations (the change of flow regimes being due either to a sharp change of the mean-slope topography or to a sharp change of basal boundary condition).

Keywords: Thin film, shallow water, lubrication, multi regimes, power law, free surface, geophysics, glaciers

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1. Introduction

This paper aims at deriving mathematical models for free-surface thin layer flows, viscoplastic (power-law rheology) with general basal boundary conditions (from no-slip to pure slip through friction) on varying basal topographies, hence multi-regimes. The study focuses on geophysical flows hence the surface tension effects are not examined.

The primitive equations considered and the employed perturbation method limit the model validity to laminar flows. A large number of geophysical flows involve viscoplastic rheology i.e. power-law fluids, from lava flows (see e.g. [1] and references therein) to mountains glaciers (inertia negligible), through ice-sheet flows (Antarctica, Greenland) (see e.g. [2] and references therein). (These two types of geophysical flows are strongly thermal dependent; nevertheless, given a temperature distribution, both present viscoplastic rheology which can be modeled by a power-law strain-stress relationship). Another large field of applications within the range of geophysical and industrial flows are paste and granular flows, from paints to debris avalanches, see e.g. [3] (nevertheless, the latter often presents a threshold at vanishing shear rate).

For Newtonian rheology and with flat bottom, gravity driven thin flows have been widely explored. The simplest model for Newtonian thin flows resulting from lubrication and shallow theory is the one-equation model derived in [4], a first order viscous model with surface tension and adherence at bottom. It provides consistent models in the limit of long wave asymptotic (thin film approximation); but for large value of the Reynolds number, the Benney equation solution tends to blow up in amplitude in finite time. Two-equation models (the shallow-water system in the newtonian case) do not present this drawback. For detailed derivations of one-equation and two-equation models, in the Newtonian case with a flat bottom and adherence (no slip), the reader can refer to [5, 6] and references therein.

In [7], a shallow-water model with asymptotically vanishing basal friction on a flat bottom, is derived up to order 2. It has been extended in two space dimensions and varying topography in [8]. Also, [9] presents models up to order 2 with basal Navier type friction law, using the methodology of [10].

On varying basal topography using a curvilinear coordinate system, let us cite [11] where a two-dimensional depth-integrated model for granular avalanches (Mohr-Coulomb law) is derived. In [12], a 1d model relaxing restrictions on the topography of the classical Saint-Venant and Savage-Hutter models (granular material) is presented. In [13], a 2d shallow water model, not depending on the parametrisation of the surface and including the case of a basal Coulomb friction law (with an asymptotically vanishing friction coefficient) is derived. In [14], similar derivations of 1d shallow-water models on varying topography but with no-slip condition at bottom are derived; the coordinate system used is the curvilinear coordinate system too. In [15] , a 1d shallow viscoelastic fluid model (Oldroyd's model), with an asymptotically vanishing viscosity and a finite relaxation time, on a varying topography is derived.

In the non-Newtonian rheology cases, models have been first elaborated empirically. Starting from the standard shallow-water equations, the authors developed extra empirical terms modeling the non-linear viscous term in the averaged momentum equation, or the friction term, see e.g. [16, 17].
In the particular case of power-law fluids with flat bottom and no-slip condition, starting from the Navier-Stokes equations, a one-equation model (Benney's type) is derived in [18]. The corresponding two-equation model has been written in [19], and even extended to yield stress fluids (Bingham's law). Another approach for deriving two-equation models was developed in [20, 21] in a weak inequality formulation. This method has been used to derive shallow models for both Bingham and Herschel-Bulkley law and vanishing friction at bottom.

Observe that all the asymptotical models derived in the literature are mono-regime in the sense the model tackles one boundary condition only: either no-slip or an asymptotically vanishing friction (Navier type law with a friction coefficient vanishing asymptotically like the geometrical ratio).
In a geophysical flow context, generalized newtonian fluids and multi-regime flows are common. Given the rheology (eg. power-law with or without a threshold), deriving an unique / unified asymptotic model de-

scribing all observed regimes remains a challenge, see e.g. [26–31]. In the particular case of glaciers flows (power-law rheology, inertia negligible), the most famous models are the Shallow Ice Approximation (SIA) if adherence at bottom (or high friction), see eg. [22–24], and the Shallow-Shelf Approximation (SSA) if pure slip at bottom, see eg. [25]. For more details, see e.g. [26, 27] and references therein.

In the present study, the asymptotic derivations remain valid for any boundary conditions at bottom (from no slip to pure slip through moderate friction, though at the price of balancing small friction by small mean slope), hence leading to "unified" / multi-regime expressions and models. All the field expressions (velocity, pressure, constraint components) are explicitly derived. Next, from the mass depth-averaged equation, one-equation models (lubrication like) are obtained. Next, from the momentum depth-averaged equation, few two-equation models (shallow-water / Saint-Venant like) are derived. All the present derivations are new since they remain valid for the whole range of friction amount at bottom, and for any varying topography (with some limitations between the slope and the friction coefficient). They lead to the so-called *unified models*. To do so, a classical perturbation method is employed; it is defined in a fixed-point form following the approach presented in [19, 32, 33]. First, all exact field expressions are formally derived at order 0, see (50) in Section 6.3, then at order 1, see eg. (53) in Section 7. Next, one-equation models (lubrication like), consistent with the Navier-Stokes free-surface flow equation / power-law rheology, can be derived classically from the averaged mass equation, see Eqn (58) and (61). They are derived both at order 0 and at order 1(the order definition is clarified later). It is shown that all classical models from the literature are recovered if considering the same assumption (eg flat bottom, newtonian case) or the same regime. The calculations enhance a natural limitation of the unified expressions and models: a vanishing friction case (pure slipe case) is possible with a vanishing mean-slope only, see (29) and the discussion in Remark 14. Next, few two-equation models (shallow-water / Saint-Venant like), see e.g. Eqn (67), (74) and (79), are derived and discussed. These two-equation models remain globally order 0 if few terms (depending on the regime considered) at derived at order 1. (Recall that two-equation models contain more information than one-equation models, see e.g. the discussion led in Section 9.2. Again, it is shown that models from the literature are recovered if considering the same assumption (eg flat bottom, newtonian case), the same boundary condition at bottom (either adherence or asymptotically vanishing friction), the same asymptotical order (in the geometrical ratio ε), the same regime (Regime C). A gradually varied flow version (like it is classically done in hydraulic), see Eqn (68), is proposed and discussed in terms similar than in hydraulic.

Concerning the basal topography representation, the curvilinear coordinates could have been chosen like in [13, 14, 17], but we chose instead easier coordinate system since in a geophysical context (on the contrary to experiments in laboratory) the topography is badly known and multi-scale. Then, in operational computational models, the topography parametrisation has to be adapted to the quality and density of the data available, and a detailed analytical description is generally not possible.

In the present study, all derivations of the solutions and models are presented either in a mean slope coordinate system with local variations (mean slope may be large) or using the Prandtl's shift, [34], see also [35, 36]. In the Prandlt coordinate system, the free surface is flatten and the terms related to the topography in the equations can be developed, see AppendixA.
The mean coordinate system is more restrictive since the existence of a meaningful mean-slope is required (with small local variations of topography), but equations are easier read. If using the Prandtl shift (instead of the mean slope coordinate system), then the asymptotic solutions remain the same, excepted the vertical velocity w, but it is not used in the depth-integrated models. A by-product of this remark is that all expressions and models remain valid for any topography variations, and even if the existence of a mean slope (with small local topography variations) cannot be clearly defined.

Concerning the well-known non-physical singularity of the deviatoric stress at free surface for shear-thinning fluids (the apparent viscosity vanishes at the free surface hence the tensor component tends to infinity, see e.g. [28, 37, 38]) does not appear in the present shallow models since all asymptotic expansions are developed at order 1 both in regime *and* in the geometrical ratio ε. This question is discussed in Section

10.

The present modeling strategy is as follows. The geometrical ratio $\varepsilon = \frac{H^*}{L^*}$ where H^* is a characteristic film thickness and L^* a characteristic streamwise length is classically defined; it is assume to be small hence shallow flows. The dimensionless Navier-Stokes equations are first written in a fixed-point like form, following the technic exposed in [19, 32, 33]. Next, the following dimensionless parameters appear naturally: $\beta = \varepsilon Re$ (balancing the inertial term), $\alpha = \varepsilon \frac{Fr^2}{Re}$ (balancing the viscous term) and $\delta = \varepsilon \frac{Re}{Fr^2}$ (balancing the pressure term). Then, the three regimes considered are as follows:

- Regime A: β, α small and $\delta = O(1)$;
- Regime B: β, δ small and $\alpha = O(1)$;
- Regime C: the three parameters small.

This last regime is a direct consequence of the two previous ones since all three parameters are small. Furthermore, it includes the case β larger ($\beta = \sqrt{\varepsilon}$). This last case remains a viscous regime (it could be denoted as a Regime D). Regime C is the regime implicitly considered in the literature, see Section 11.
The asymptotic expansions are developed with respect to at least two of these three parameters (with the assumption that ε small).
All regimes include potentially (non-linear) friction boundary conditions at bottom, from no-slip to pure slip (but potentially respecting the mean slope / friction coefficient suitability mentioned previously). The present regime classification can be related to the review classification presented in [39] where the regimes considered are: inertial (which is not possible in the present approach), viscous (Regime B with small slope λ), visco-inertial (limit case of Regime A), nearly steady-uniform (Regime C with large slope λ).
Finally, let us remark that these regimes can be re-interpreted in terms of the dimensionless numbers Re and Fr, some typical cases are presented in Section 2.3.2.

The paper is organized as follows. In Section 2, the dimensionless Navier-Stokes power-law free surface equations, with friction condition at bottom and varying topography, are presented in a fixed point form following [19, 32]. This formulation makes appear naturally the dimensionless numbers β, α, δ introduced above. In the present derivations, the velocity scaling leads naturally to a stress tensor scaling. In Section 3, the reference flows corresponding to different boundary condition at bottom are presented. The reference flow definitions are important since all the asymptotic fields are calculated as perturbations of these reference flows. The perturbation expansion algorithm used to calculate the explicit expressions of all fields $(\sigma, \mathbf{u}, \mathbf{p})$, up to order 1 or 2 (with respect to the small dimensionless numbers above and/or ε) is presented in Section 4. Formal error estimates of the calculations are presented in Section 5, thus validating the asymptotic derivations. The error estimates show that the global model error can be dominated by the basal friction model error (which is a law stated a-priori). The explicit expressions of all fields are derived at order 0 in Section 6, and at order 1 in Section 7. In Section 8, by injecting classically the discharge expression (valid for all regimes) into the depth averaged mass equation, the one-equation models (lubrication) are obtained both at order 0 and at order 1. The 0th order one-equation model includes distinct terms for Regime A and Regime B. The 1st order model contains extra terms in $\beta, \beta\delta$. Few particular cases, such as the Newtonian or pure slip cases, are detailled. In Section 9, a preliminary two-equation model is stated. The model is preliminary since it is not fully consistent yet with the primitive model (Navier-Stokes, power-law, free-surface). Then a gradually varied flow equation can be derived and analyzed following the hydraulic like model derivations. In order to reach a consistent two-equation model, few field expressions must be derived at order 1; it is done in Section 10. These extra expansions result to consistent classes of two-equation models. The first natural version of the two-equation model (in conservative form) turns out to be linearly ill-posed, see Section 10.3. Then, a quasi-linear form of this unified two-equation model (which could be called a three-equation model) is proposed since it is linearly well-posed, see Section 10.4. This last model version is the correct and complete form of these newly derived multi-regime model families.
Few particular cases are detailed (pure slip case, newtonian, mono regime given). In Section 11, comparisons

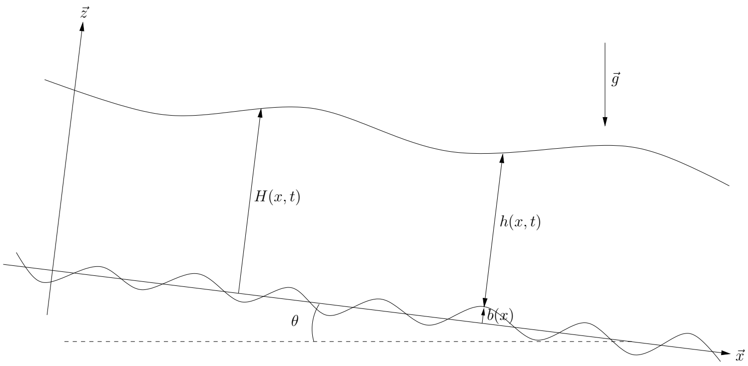


Figure 1: Geometry and notations. Mean-slope coordinate system. $H(x,t) = h(x,t) + b(x)$.

with the existing models in literature are made; it is demonstrated that the present models include each existing model presented if considering the same assumption.
In Section 12, the unified steady-state one-equation model order 0 is solved numerically in the cases of two regime thin flows with sharp regime transitions. The numerical results demonstrate the ability of the unified model to represent accurately multi-regime flows in presence of sharp, continuous transition; the change of regime being due to either a change of bottom boundary condition at bottom, or a change of topography (the mean slope in this case). A conclusion is proposed in Section 13. In AppendixA, the Prandlt shift is detailed; in AppendixB, some quite long first order coefficient expressions which appear in the discharge expression are presented; while those appearing in the consistent two-equation model are detailed in AppendixC.

2. Primitive equations

In this section the dimensionless Navier-Stokes equations for power-law fluids with free surface and with a general power law friction law at bottom, are written. The scaling of the velocity naturally defines the scaling of the stress tensor. In other respect, the present scalings are compared to other ones from the literature.

2.1. Flow equations and dimensionless parameters

The fluid is assumed to be isotropic, non-Newtonian with a power-law type rheology. The power law is employed for a large range of fluid flows. The topography can be non constant. The coordinate system (x, z) is the "mean-slope" coordinate system, see figure 1. Later, it is shown that defining a mean slope is not a necessary condition; using the Prandtl's shift, the results obtained remain valid for arbitrary large topography variations.

Here, H is the fluid elevation, b the topography elevation, $h = (H - b)$ the fluid depth, $\mathbf{u} = (u, w)$ the velocity, π the pressure and θ the bottom mean slope, see figure 1. Remark that if $\theta = 0$, then (u, w) denotes the horizontal-vertical velocities components. $\vec{g}$ is the gravity source term and a is a mass balance source term at the free surface (it may depend on space and time $a = a(x, t)$).

2.1.1. Flow equations

A two dimensional flow described by the Navier-Stokes equations is considered:

$$\begin{array}{rl} \nabla \cdot \mathbf{u} & = 0 \\ \varrho \left(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) & = \nabla \cdot \sigma - \nabla \pi + \varrho \mathbf{g} \end{array} \quad \text{in } \Omega_t \times]0, T[\tag{1}$$

with $(x, z) \in \Omega_t = \left\{ (x, z) \in \mathbb{R}^2 | b(x) \leq z \leq H(x, t) \right\}$.

Here, $\mathbf{u} = (u, w)^T$ is the fluid velocity (u the streamwise and w the cross-stream component), π the pressure, $\Sigma = \sigma - \pi \mathrm{Id}$ the full stress tensor with $\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{pmatrix}$ the deviatoric stress tensor, ϱ the fluid density and $\mathbf{g} = (g \sin\theta, -g\cos\theta)^T$ the gravity.

2.1.2. Rheology: power-law fluid

The generalized Newtonian fluid for which the shear stress τ is related to the deviatoric stress tensor σ by the Norton-Hoff Law (a viscoplastic law) is considered. This law is also known as the Ostwald-de Waele law. The Norton-Hoff law is considered in other fields such as glaciology, frozen soil mechanics or lithospheric materials.

The stress tensor Σ, is defined by:

$$\Sigma(\mathbf{u}, \pi) = \sigma(\mathbf{u}) - \pi \mathrm{Id} = 2\mu D(\mathbf{u}) - \pi \mathrm{Id} \tag{2}$$

where $\sigma(\mathbf{u}) = (\sigma_{ij})_{1 \leq i,j \leq 2}$, is the deviatoric stress tensor, μ is the apparent viscosity and $D(u)$ is the symmetric rate of strain tensor:

$$D(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right) = \begin{pmatrix} \partial_x u & \frac{\partial_z u + \partial_x w}{2} \\ \frac{\partial_z u + \partial_x w}{2} & \partial_z w \end{pmatrix} \tag{3}$$

For a non-Newtonian fluid, the viscosity μ is a scalar function of the rate of strain tensor through the second invariant of $D(\mathbf{u})$:

$$I_2 = \|D(\mathbf{u})\|_F^2 = \sum_{i,j} D_{ij} D_{ji} = \partial_x u^2 + \frac{1}{2}(\partial_z u + \partial_x w)^2 + \partial_z w^2$$

Since the flow is incompressible, the following relation holds:

$$\partial_x u = -\partial_z w \Leftrightarrow \sigma_{xx} = -\sigma_{zz}$$

and then the expression of the second invariant can be simplified. The strain rate $\dot{\gamma}$ is defined as:

$$\dot{\gamma} = \sqrt{2 I_2} = \sqrt{2}\, \|D(\mathbf{u})\|_F$$

where$\|D(\mathbf{u})\|_F = \sqrt{2\partial_x u^2 + \frac{1}{2}(\partial_z u + \partial_x w)^2}$. The power-law expresses the apparent viscosity using the strain rate:

$$\mu = \mu_0 \dot{\gamma}^{n-1} \tag{4}$$

Remark 1. In multi-physics flows, μ_0 may depend on an extra field. Typically, for ice or lava flows, it depends on the temperature.

Finally the deviatoric stress tensor can be written as:

$$\sigma = 2\mu_0 \dot{\gamma}^{n-1} D(\mathbf{u}) = 2\mu D(\mathbf{u}) \tag{5}$$

The relation (5) can be inversed. The norm of σ writes:

$$\|\sigma\|_F = 2\mu_0 \dot{\gamma}^{n-1} \|D(\mathbf{u})\|_F = \sqrt{2}\mu_0 \dot{\gamma}^n$$

Introduce the scalar σ_e for the effective stress defined as follows:

$$\sigma_e = \frac{1}{\sqrt{2}} \|\sigma\|_F = \mu_0 \dot{\gamma}^n = \sqrt{\sigma_{xx}^2 + \sigma_{xz}^2}$$

The shear rate using the effective stress can be written:

$$\dot{\gamma} = \left(\frac{\sigma_e}{\mu_0}\right)^{\frac{1}{n}}$$

Then, the so-called Nye-Glen's flow law (the dual law of Ostwald-de Waele's law) is obtained. The latter writes:

$$D(\mathbf{u}) = A\sigma_e^{p-1}\sigma \tag{6}$$

with:

$$A = \frac{1}{2\mu_0^p} \qquad \mu = \mu_0^p \sigma_e^{1-p} \qquad p = \frac{1}{n} \qquad \sigma_e = \sqrt{\sigma_{xx}^2 + \sigma_{xz}^2}$$

where A is the rate factor and p the stress exponent.

Three types of fluids, depending on the power law index value, can be defined:

- pseudoplastic fluids (shear thinning fluids) for $0 < n < 1$, apparent viscosity decreases when shear stress increases (e.g. ice, lava, mud and many industrial fluids);
- dilatant fluids (shear thickening fluids) for $n > 1$, apparent viscosity increases with shear rate (less common fluids);
- newtonian fluids for $n = 1$, the shear stress is proportional to the shear rate (we get $\mu = \mu_0$).

The system of Navier-Stokes equations (1) expanded reads:

$$\begin{array}{rl} \partial_x u + \partial_z w = 0 & \\ \varrho\left(\partial_t u + u\partial_x u + w\partial_z u\right) = \partial_x\left(\sigma_{xx}\right) + \partial_z\left(\sigma_{xz}\right) - \partial_x \pi + \varrho g \sin\theta & \text{in } \Omega_t \times]0,T[\\ \varrho\left(\partial_t w + u\partial_x w + w\partial_z w\right) = \partial_x\left(\sigma_{xz}\right) + \partial_z\left(\sigma_{zz}\right) - \partial_z \pi - \varrho g \cos\theta & \end{array} \tag{7}$$

2.1.3. Boundary conditions

- At bottom, where $(x,z) \in \Gamma^b = \left\{(x,z) \in \mathbb{R}^2 | z = b(x)\right\}$, one of both cases is considered:
  - no-slip condition (adherence):
  $$\mathbf{u}_{|z=b} = 0$$
  - friction condition. A power-law friction condition (also known as Weertman-type friction law) is considered:
  $$\left\{\begin{array}{c} \mathbf{u}\cdot\mathbf{t}_b = -C_d\left|\sigma\cdot\mathbf{n}_b\cdot\mathbf{t}_b\right|^{p-1}\sigma\cdot\mathbf{n}_b\cdot\mathbf{t}_b \\ \mathbf{u}\cdot\mathbf{n}_b = 0 \end{array}\right. \quad \text{in } \Gamma^b \times]0,T[$$
  with C_d the basal slipperiness (the minus sign causes C_d to be positive as the normal to the bottom is pointing outward). Vectors $\mathbf{n}_b$ and $\mathbf{t}_b$ are respectively the outward normal and the tangent to the bottom. Note that this general expression contains the no-slip condition when the slip coefficient C_d vanishes.

  Remark 2. If there exists a heterogeneous layer between the solid at bottom (eg bedrock) and the fluid homogeneous fluid, then the friction law exponent may differ from the bulk rheology exponent.

- At the top (liquid-gas interface), where $(x,z) \in \Gamma_t^s = \left\{(x,z) \in \mathbb{R}^2 | z = H(x,t)\right\}$, the following equations hold:
  - continuity of the velocity:
  $$\partial_t H + u(t,H)\,\partial_x H = w(t,H) + a \text{ in } I \times]0,T[$$
  where I is a x-interval and a is a mass balance term given.
  - continuity of the fluid stress (without taking into account capillary effects):
  $$(\sigma - \pi \mathrm{Id})\,\mathbf{n} = 0 \text{ in } \Gamma_t^s \times]0,T[\tag{8}$$
  Vectors $\mathbf{n}$ and $\mathbf{t}$ are respectively the normal and the tangent to the free surface. By projecting (8) on $\mathbf{n}$ and $\mathbf{t}$ and with some computations, the condition at free surface writes:
  $$\left\{\begin{array}{l} \sigma_{xz} = 2\sigma_{xx}\frac{\partial_x H}{1-\partial_x H^2} \\ \pi = -\sigma_{xx}\frac{1+\partial_x H^2}{1-\partial_x H^2} \end{array}\right. \quad \text{in } \Gamma_t^s \times]0,T[$$

Remark 3. Surface tension effects is not considered here for a sake of simplicity, also it is not necessary for geophysical flows. Furthermore, it is easy to take it into account in all the following calculations. In other respect, recall it has already been done for Regime C in [19].

2.2. Scaling

Let (H^*, U^*, P^*) be the characteristic values of the flow.

Remark 4. The characteristic values of the flow will be set as the mean values of the reference flows presented in the next subsection.

The dimensionless variables can be defined:

$$\begin{array}{cccc} x = L^*\tilde{x} & z = H^*\tilde{z} & h = H^*\tilde{h} & b = H^*\tilde{b} \\ t = \frac{L^*}{U^*}\tilde{t} & u = U^*\tilde{u} & w = W^*\tilde{w} & \pi = P^*\tilde{\pi} \end{array} \tag{9}$$

The mass equation (incompressibility constraint) suggests to define: $W^* = \frac{H^*}{L^*}U^*$.

The pressure can be scaled by the hydrostatic pressure: $P^* = \rho g H^* \cos\theta$.

The geometric scaling parameter is defined by:

$$\varepsilon = \frac{H^*}{L^*} \tag{10}$$

Remark 5. The present mean-slope coordinate system prevents to consider large variations of the local topography $\partial_x b$. Indeed, large variations would induced large variations of the normal component of the velocity w, which is $O(\varepsilon)$ due to the mass equation. Nevertheless, in Section AppendixA, it is shown that using the Prandtl's shift all the results remain valid for potentially large arbitrary topography variations.

Define the so-called equivalent viscosity:

$$\mu_e = \mu_0 \left(\frac{U^*}{H^*}\right)^{n-1} \tag{11}$$

It defines the natural scaling of the viscosity norm.

Remark 6. The scaling of the deviatoric stress tensor σ is a direct consequence of the scaling of the velocity (9). It gives $\sigma = \mu_e \frac{U^*}{H^*}\tilde{\sigma}$ with:

$$\tilde{\sigma} = \begin{pmatrix} \varepsilon\tilde{\sigma}_{xx} & \tilde{\sigma}_{xz} \\ \tilde{\sigma}_{xz} & -\varepsilon\tilde{\sigma}_{xx} \end{pmatrix}$$

Similarly, the following dimensionless strain rate are naturally obtained:

$$\tilde{\dot{\gamma}} = \sqrt{4\varepsilon^2 \partial_{\tilde{x}}\tilde{u}^2 + \left(\partial_{\tilde{z}}\tilde{u} + \varepsilon^2\partial_{\tilde{x}}\tilde{w}\right)^2} \tag{12}$$

Note that: $\tilde{\dot{\gamma}} = O(1)$. The constitutive law writes:

$$\tilde{D} = \frac{1}{2}\tilde{\sigma}_e^{p-1}\tilde{\sigma}$$

In its dimensionless form the apparent viscosity reads:

$$\mu = \mu_e \tilde{\sigma}_e^{1-p}$$

with $\tilde{\sigma}_e$defined by $\tilde{\sigma}_e = \sqrt{\tilde{\sigma}_{xz}^2 + \varepsilon^2\tilde{\sigma}_{xx}^2}$. The apparent viscosity writes equivalently:

$$\mu = \mu_e \tilde{\dot{\gamma}}^{n-1}$$

with $\tilde{\tau}$ defined by (12).

The mass balance source term scaling is given by: $a = \varepsilon U^* \tilde{a}$. Finally, the basal slipperiness scaling and the mass balance source term scaling are given by:

$$C = \mu_0^p \frac{C_d}{H^*} \tag{13}$$

where $\mu_0^p C_d$ has the dimension of a length. Thus it is called the slip length and it characterizes the fluid/solid interaction. Let us recall that $C \to \infty$ corresponds to a pure slip at bottom while $C \to 0$ corresponds to the adherence case (no-slip).

The equivalent viscosity defined by (11) leads to the standard Reynolds number definition as follows:

$$Re = \frac{\varrho U^* H^*}{\mu_e} \tag{14}$$

The Froude number is defined as follows:

$$Fr = \frac{U^*}{\sqrt{gH^* \cos\theta}} \tag{15}$$

2.3. Small numbers and resulting regimes

2.3.1. The dimensionless numbers

The Navier-Stokes equations will be written as a fixed-point like form, following the method exposed in [19, 32], then the dimensionless parameters which appear naturally, see Eqn(20), are:

$$\boxed{\beta = \varepsilon Re, \ \ \alpha = \varepsilon\gamma \ , \ \ \delta = \frac{\varepsilon}{\gamma}} \tag{16}$$

With:

$$\gamma = \frac{Fr^2}{Re} \tag{17}$$

The dimensionless number β is the weight coefficient corresponding to the inertial term, α is the weight coefficient corresponding to the viscous term and δ the weight coefficient corresponding to the pressure. Let us remark that in physical variables, the parameter γ writes:

$$\gamma = \frac{\mu_0}{\rho g H^* cos\theta} (\frac{U^*}{H^*})^{\frac{1}{p}}$$

Note that the following relationship holds:

$$\varepsilon^2 = \alpha\delta \tag{18}$$

The dimensionless gravity source term due to the bedrock mean-slope is defined by:

$$\boxed{\lambda = \frac{1}{\gamma} \tan(\theta) = \sqrt{\frac{\delta}{\alpha}} \tan(\theta)} \tag{19}$$

And let us recall the dimensionless slip parameter given by (13).

In all the sequel, it is assumed

Assumption 1.

- *the geometrical parameter ε is small,*
- *the dimensionless parameters (α, β, δ) are either small or at most $O(1)$.*
- *the flows considered are perturbation of a steady-state uniform flow (the references flows defined in next section).*

The asymptotic expansions derived below will depend on the dimensionless parameters (α, β, δ), the mean-slope gravity term λ and the slip coefficient C.

	ε	Fr	Re	β	α	δ	γ	λ	C
"Ice-sheet"	10^{-3}	3.10^{-8}	3.10^{-15}	10^{-18}	10^{-4}	10^{-2}	10^{-1}	from 0 to 10^{+1}	from 0 to O(1)
"Ice-streams"	10^{-2}	10^{-6}	10^{-13}	10^{-14}	10^{-1}	10^{-3}	10^{+1}	from 0 to 10^{-1}	from O(1) to large

Table 1: Example in glaciology. Orders of magnitude of dimensionless parameters upon the flow regime considered.

2.3.2. Definition of the three flow regimes considered

In the present study, asymptotic models are derived for the following three laminar viscous flow regimes:

- Regime A: β small, α small while $\delta = O(1)$.
- Regime B: β small, δ small while $\alpha = O(1)$.
- Regime C: β small, α small and δ small.

Note that Regime C is a direct consequence of Regime A and B since all three parameters are small. Also, a moderate inertial regime is included in the present viscous regimes since β can set to $\sqrt{\varepsilon}$ (i.e. $Re = \frac{1}{\sqrt{\varepsilon}}$). Such a regime is called the visco-inertial regime in the classification made in [39].

Remark 7. Since the asymptotic models will be derived as perturbations of steady-state uniform flows (the reference flows defined in next section), it will appear a relationship between λ and C, see Eqn (28). In consequence, the regimes (A, B or C) defined above may be related to the friction amount and the mean slope value; this feature is clarified in next section.

Let us give examples of the three regimes in terms of the dimensionless numbers (Re, Fr).
First let us consider the case $Fr \approx 1$. Then, $\delta \approx \beta \approx \varepsilon Re$, $\alpha \approx \varepsilon Re^{-1}$. Then,

- if $Re \approx 1$, this corresponds to Regime C.
- if $Re \approx \sqrt{\varepsilon}$, this corresponds to Regime B.
- if $Re \approx \dfrac{1}{\sqrt{\varepsilon}}$, this corresponds to Regime C.

Next, let us consider the case $Re \approx 1$. Then, $\beta \approx \varepsilon$, $\alpha \approx \varepsilon Fr^2$, $\delta \approx \varepsilon Fr^{-2}$. Then,

- if $Fr \approx 1$, same case as above, it corresponds to Regime C.
- if $Fr \approx \sqrt{\varepsilon}$, this corresponds to Regime A.
- if $Fr \approx \dfrac{1}{\sqrt{\varepsilon}}$, this corresponds to Regime B.

An example : ice flow regimes. In ice flows, different regimes are observed. Let us give orders of magnitude of the corresponding dimensionless parameters. The most common value for the rheological exponent is $p = 3$. Let us consider for the viscosity μ_e a "medium value" ($\mu_e \approx 10^{14} - 10^{15}$) (recall its value depends strongly on the temperature, see e.g. Greve and Blater [2]).We present below two typical regimes observed in Antarctica. First, the "ice-sheet" regime which represents typical large scale orders of magnitude of the whole ice-sheet. Second the "ice-stream" regime which represents typical orders of magnitude small scale of streams flows either (in Antarctica but also in mountains such as the Himalaya or the Alps). Since ice is extremely viscous, β is negligible for both regimes.

For the so-called “ice-sheet” regime, α is negligible compared to δ. At contrary for the so-called “ice-streams” regime, δ is negligible compared to α. In the glaciology context, the “ice-sheet” regime would correspond to Regime A with no-slip at bottom (the "slip" coefficient C defined by (13) vanish). While the “ice-stream” regime would correspond to the Regime B with variable friction boundary condition at bottom (C would vary from medium to large values).

Remark 8. A transition between Regime A and Regime B is possible only if all parameters are small, i.e. in the particular case of Regime C. It is important to remark that it is the only possible asymptotic transition between the two regimes. In fact others transitions involve non-small values of α and β, which means the geometrical parameter $\varepsilon = \sqrt{\alpha\delta}$ cannot be small, and thus there is no asymptotic expansion available. Nevertheless, in such a case, it is still possible to use the full Navier-Stokes equations to connect the two regimes, since we fully know the velocity and the pressure fields from the asymptotic solution (see their expressions in next sections).

Remark 9. The time scale implicitly considered above is $t^* = \frac{L^*}{U^*}$. Therefore, in terms of flow dynamics, the asymptotical expansions developed in all the sequel remain valid while t^* remains $O(1)$. Typically, in the ice flow example detailed above, this time scale corresponds to one century (which is already quite a long time range).

In the sequel, the way to obtain the asymptotic models will be different depending on the regime considered, and the boundary conditions considered at bottom (either adherence or friction, from large to vanishing).

2.4. Dimensionless primitive equations

The dimensionless Navier-Stokes free-surface equations are written such that the quantities $(\partial_z\sigma_{xz}, \partial_z u, \partial_z\pi, \partial_z w, \sigma_{xx})$ are explicitly expressed. In all the sequel, the symbol $\tilde{\cdot}$ on all variables is skipped since all variables are dimensionless.

2.4.1. Equations in the bulk

From Equation (7) the dimensionless form of the mass conservation equation and the momentum equations are written. The deviatoric stress tensor components are kept as unknowns. These unknowns are related to the velocity unknowns (u, w). By reorganizing the terms, the equations in the bulk Ω_t and for any time, are written as follows:

$$\left\{\begin{array}{rl}\partial_z\sigma_{xz} & = -\lambda + \delta\partial_x\pi + \beta\left(\partial_t u + u\partial_x u + w\partial_z u\right) \\ & \quad -\alpha\delta\partial_x\sigma_{xx} \\ \partial_z u & = \sigma_e^{p-1}\sigma_{xz} - \alpha\delta\partial_x w \\ \partial_z\pi & = -1 - \alpha\partial_z\sigma_{xx} + \alpha\partial_x\sigma_{xz} \\ & \quad -\alpha\beta\left(\partial_t w + u\partial_x w + w\partial_z w\right) \\ \partial_z w & = -\partial_x u \\ \sigma_{xx} & = 2\sigma_e^{1-p}\partial_x u\end{array}\right. \qquad \text{in } \Omega_t\times]0,T[\tag{20}$$

with $\sigma_{zz} = -\sigma_{xx}$ and the effective stress given by: $\sigma_e = \sqrt{\sigma_{xz}^2 + \alpha\delta\sigma_{xx}^2}$.

Remark 10. In the system of equations (20), the case $\beta = 0$ corresponds to the *a-priori* approximation of a quasi-static Stokes flow. This assumption can be made a-priori if modeling very viscous fluid flow, typically if modeling ice flows. Furthermore, if considering model order 1 in α or δ, and order 1 in the aspect ratio ε (recall $\varepsilon^2 = \alpha\delta$), the dimensionless primitives equations write:

$$\left\{\begin{array}{rl}\partial_z\sigma_{xz} & = -\lambda + \delta\partial_x\pi \\ \partial_z u & = |\sigma_{xz}|^{p-1}\sigma_{xz} \\ \partial_z\pi & = -1 - \alpha\partial_z\sigma_{xx} + \alpha\partial_x\sigma_{xz} \\ \partial_z w & = -\partial_x u \\ \sigma_{xx} & = 2|\sigma_{xz}|^{1-p}\partial_x u\end{array}\right. \qquad \text{in } \Omega_t\times]0,T[\tag{21}$$

2.4.2. Boundary conditions

In its dimensionless form, the free surface dynamics equation writes:

$$\partial_t H + u\,(t, H)\,\partial_x H = w\,(t, H) + a \text{ in } I \times]0, T[\tag{22}$$

and the continuity of the stress tensor writes:

$$\left\{ \begin{aligned} \sigma_{xz} &= 2\alpha\delta\sigma_{xx}\frac{\partial_x H}{1 - \alpha\delta\partial_x H^2} \\ \pi &= -\alpha\sigma_{xx}\frac{1 + \alpha\delta\partial_x H^2}{1 - \alpha\delta\partial_x H^2} \end{aligned} \right. \quad \text{in } \Gamma_t^s \times]0, T[\tag{23}$$

At bottom the friction condition writes:

$$\left\{ \begin{aligned} u &= C\frac{\left|\sigma_{xz}\left(1 - \alpha\delta\partial_x b^2\right) - 2\alpha\delta\sigma_{xx}\partial_x b\right|^{p-1}}{\left(1 + \alpha\delta\partial_x b^2\right)^{p+\frac{1}{2}}}\left(\sigma_{xz}\left(1 - \alpha\delta\partial_x b^2\right) - 2\alpha\delta\sigma_{xx}\partial_x b\right) \\ w &= u\partial_x b \\ &\text{in } \Gamma^b \times]0, T[\end{aligned} \right. \tag{24}$$

In next section, the continuous transition from $C = 0$ (adherence) to $C \to \infty$ (pure slip) is considered.

Remark 11. At order 1 in α or δ, and order 1 in the aspect ratio ε (recall $\varepsilon^2 = \alpha\delta$), the boundary conditions write:

$$\sigma_{xz} = 0\ ;\quad \pi = -\alpha\sigma_{xx} \quad \text{in } \Gamma_t^s \times]0, T[\tag{25}$$

$$u = C\left|\sigma_{xz}\right|^{p-1}\sigma_{xz}\ ;\quad w = u\partial_x b \quad \text{in } \Gamma^b \times]0, T[\tag{26}$$

Remark 12. In case of a flat-bottom ($\partial_x b = 0$), and if choosing the corresponding mean-slope coordinate system, the friction boundary condition writes:

$$\left\{ \begin{aligned} u &= C\left|\sigma_{xz}\right|^{p-1}\sigma_{xz} \\ w &= 0 \end{aligned} \right. \quad \text{in } \Gamma^b \times]0, T[\tag{27}$$

The limit case pure slip is $C \to \infty$.

3. Reference flows : steady-state, uniform

We define the reference flow as the stationary flow, uniform in x, solution of the zero-th order equations. It is equivalent to set: $b = 0$, $h = H = 1$ and $\alpha = \beta = \delta = 0$.

In other respect, the calculations below show few limitations : the gravitational viscous flow can not be a "plug flow" if the mean slope θ is not vanishing. Also, Regime A with high slip will be proved to be possible only in case of very small mean-slope.

The system of equations (20) with the boundary conditions (23) and (27) give the uniform flow solution:

$$\left\{ \begin{aligned} \sigma_{xz} &= \lambda\,(1 - z) \\ \pi &= (1 - z) \\ u &= C\lambda^p + \tfrac{\lambda^p}{p+1}\left(1 - (1 - z)^{p+1}\right) \\ w &= \sigma_{xx} = 0 \end{aligned} \right.$$

The stress tensor is a linear function of z. The pressure is hydrostatic and the vertical profile of u is a Poiseuille-like flow for $p = 1$ (Newtonian case).

Remark 13. Let us recall that, for $p > 1$, close to the free surface (i.e. $z = 1$) the stress tensor component vanish: $\sigma_{xz} \underset{z\to 1}{\longrightarrow} 0$, therefore the apparent viscosity of the power-law fluid tends towards infinity: $\mu = \mu_e \sigma_e^{1-p} \underset{z\to 1}{\longrightarrow} \infty$. In other words there is a plug like flow boundary layer close to the free surface.

For a geometrical parameter θ given, once settled either the Reynolds number Re or the Froude number Fr, we obtain the second one by setting λ since : $\lambda = \dfrac{Re}{Fr^2} \tan\theta$.

Then, the setting of the reference velocity U^* (equal to the surface velocity $u_{|z=H}$ or the mean velocity $\bar{u} = \int_b^H u(z)\,\mathrm{d}z$ for example) implies a relation between λ and C:

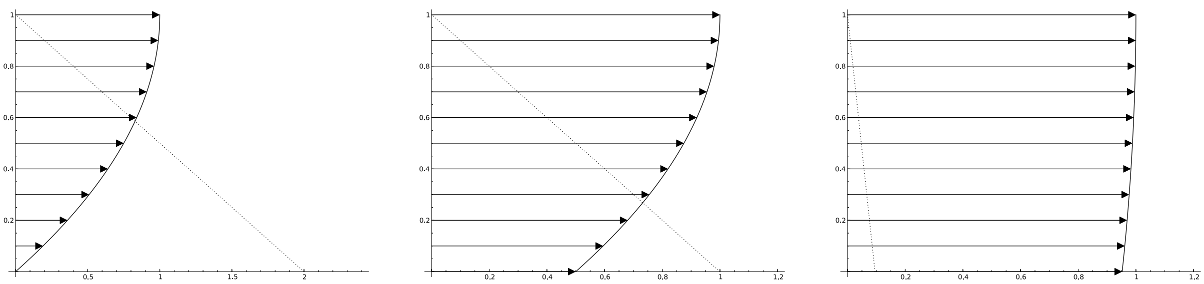
$$\lambda\left(U^* = u_{|z=H}\right) = \left(\frac{p+1}{C(p+1)+1}\right)^{\frac{1}{p}} \quad ; \quad \lambda\left(U^* = \bar{u}\right) = \left(\frac{p+2}{C(p+2)+1}\right)^{\frac{1}{p}} \tag{28}$$

As a first consequence, the pure slip case $C \to \infty$ corresponds necessarily to a vanishing mean slope in the sense :

$$\lambda \underset{C\to\infty}{\sim} \frac{1}{C^{1/p}}. \tag{29}$$

In the Newtonian case ($p = 1$), we recover the classical result that for a Nusselt film, the interface velocity and the mean velocity are given by $Re\tan\theta = 2Fr^2$ using the surface velocity as the reference velocity, or $Re\tan\theta = 3Fr^2$ using the mean velocity.

We plot the vertical velocity profiles corresponding to the three reference flows in figures 2 and 3. The reference velocity is the surface velocity. These reference flows correspond to the three cases we are interested in, which corresponds to different condition at bottom. They present a transition between the no-slip condition (a) ($C = 0$) to a pure slip condition ($C \to \infty$) through the intermediate case (b) with C at most $O(1)$.



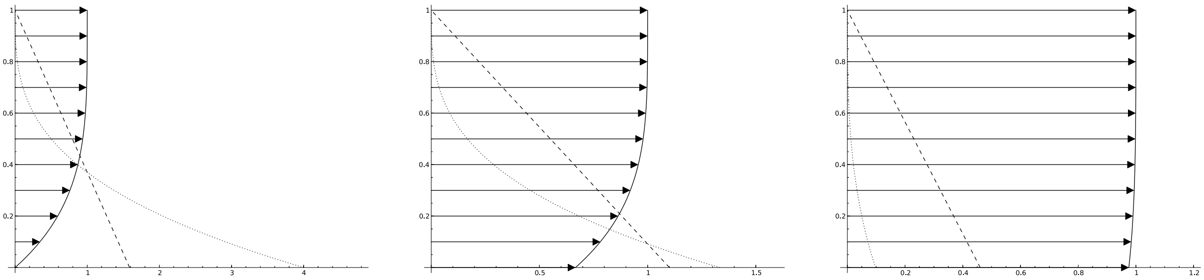
(a) $\lambda = 2$, $C = 0$ (b) $\lambda = 1$, $C = 0.5$ (c) $\lambda = \frac{2}{21}$, $C = 10$

Figure 2: Reference flows vertical velocity profile in the newtonian case ($p = 1$):
— u ; $\cdots$ $\partial_z u$

Remark 14. We can replace λ by its value (28), and passing to the limit $C \to \infty$, we obtain: $\sigma_{xz}(z) \underset{C\to\infty}{\longrightarrow} 0$ and $u(z) \underset{C\to\infty}{\longrightarrow} 1$. For shear thinning fluids, as the stress tensor vanishes, the apparent viscosity goes to infinity, meaning the flow moves as a solid plug:

$$\mu = \mu_e \|\sigma\|^{1-p} \underset{C\to\infty}{\longrightarrow} \infty \qquad (p \geq 1)$$

Therefore we will call the pure slip case $C \to \infty$ a "plug flow". In that case, we necessarily have $\lambda = 0$. As a matter of fact, in the present model a gravitational viscous flow can not be the so-called plug flow if the mean slope θ is not vanishing.



(a) $\lambda = \sqrt[3]{4}$, $C = 0$

(b) $\lambda = \sqrt[3]{\frac{4}{3}}$, $C = 0.5$

(c) $\lambda = \sqrt[3]{\frac{4}{41}}$, $C = 10$

Figure 3: Reference flows vertical velocity profile for $p = 3$
— u ; $\cdots$ $\partial_z u$; - - σ_{xz}

Remark 15. If considering a steady-state uniform flow with $h = q = 1$ (hence $\partial_x H = 0$), with flat bottom, equation (28) can be seen as a relation between the regime flow parameter γ and the geometrical parameter $\tan\theta$ (the mean slope), and writes as follows:

$$\gamma = \frac{\tan\theta}{\left(\frac{p+2}{C(p+2)+1}\right)^{\frac{1}{p}}} \tag{30}$$

Since the (dimensionless) basal slipperiness C must be positive, for shear thinning fluids ($p \geq 1$) we get:

$$\tan\theta \leq 3\gamma \tag{31}$$

Let us recall that typical value of γ in Regime A is $\gamma = \varepsilon$ and in Regime B is $\gamma = \varepsilon^{-1}$. As a consequence, Regime A with sliding is possible only in case of very small mean-slope.

For shear-thickening fluids, only equation (30) holds; and in Regime A one can have any slip coefficient for any mean-slope value.

4. Asymptotic iterates

The method to calculate the exact solutions and the asymptotic models is a follows. First, we write an "explicit expression" of the unknowns $(\sigma_{xx}, \sigma_{xz}, u_{|z=b}, u, w, \pi)$. Second, we write the Navier-Stokes system as a fixed-point like system, where the unknown is $(\sigma_{xx}, \sigma_{xz}, u_{|z=b}, u, w, \pi)$. Third, by iterating the calculations (once or twice upon the asymptotic order required), we obtain the formal higher-order expansion terms of each field. This way to iterate the asymptotic expansions, due to the third author, has been presented first in [32], next in [19] for mono-regime flows (power-law and Bingham). This iterative algorithm remains a classical perturbation method. In the present study, this way to calculate the exact solutions and the asymptotic models is adapted to the present multi-regimes context since the goal is to derive unified expressions, valid for all regimes. This feature requires to keep the right expansion terms for each regime. Then the expansion algorithm has to be adapted following the regime considered, see Section 4.3.

Let us point out that, in the newtonian case, a rigorous mathematical justification of such a asymptotic process is presented in [33]. In the case of a flat bottom and adherence at bottom, it is proved that the present iterative method gives rigorously the asymptotic solutions of the Navier-Stokes equations, thus obtaining (new) consistent Saint-Venant's type equations.

Below, we develop the calculations for the three regimes considered (regimes A, B and C). Each solution and model are obtained as a perturbation of the corresponding reference flow (see the previous section).

4.1. "Explicit expressions" of the unknowns $(\sigma_{xx}, \sigma_{xz}, u_{|z=b}, u, w, \pi)$

The stress tensor, the pressure and the velocity are obtained by solving the system of equations (20) with the boundary conditions (23) and (27). We use the Leibnitz rule to invert summations and derivatives. Then the solution satisfies the following relations:

$$\boxed{\left\{\begin{array}{rl} \sigma_{xz} & = \psi_{xz}\left(\sigma_{xx}, \pi, u, w\right) \\ \sigma_{xx} & = \psi_{xx}\left(\sigma_{xz}, \sigma_{xx}, u\right) \\ \pi & = \psi_{\pi}\left(\sigma_{xz}, \sigma_{xx}, u, w\right) \\ u_{|z=b} & = \psi_{bu}\left(\sigma_{xz}, \sigma_{xx}\right) \\ u & = \psi_{u}\left(\sigma_{xz}, \sigma_{xx}, u_{|z=b}, w\right) \\ w & = \psi_{w}\left(u_{|z=b}, u\right) \end{array}\right.} \tag{32}$$

where the functions $\psi_{\square}$ are defined as follows:

$$\left\{\begin{array}{rl} \psi_{xz}\left(\sigma_{xx}, \pi, u, w\right) & = \lambda\left(H-z\right) - \delta\partial_x\left(\int_z^H \pi \mathrm{d}z\right) \\ & \quad -\beta\left(\partial_t\left(\int_z^H u\mathrm{d}z\right) + \partial_x\left(\int_z^H u^2\mathrm{d}z\right) - uw - au_{|z=H}\right) \\ & \quad +\alpha\delta\partial_x\left(\int_z^H \sigma_{xx}\mathrm{d}z\right) \\ \psi_{xx}\left(\sigma_{xz}, \sigma_{xx}, u\right) & = 2\left\|\sigma_e\right\|^{1-p}\partial_x u \\ \psi_{\pi}\left(\sigma_{xz}, \sigma_{xx}, u, w\right) & = \left(H-z\right) - \alpha\left(\sigma_{xx} + \partial_x\left(\int_z^H \sigma_{xz}\mathrm{d}z\right)\right) \\ & \quad +\alpha\beta\left(\partial_t\left(\int_z^H w\mathrm{d}z\right) + \partial_x\left(\int_z^H uw\mathrm{d}z\right) - w^2 - aw_{|z=H}\right) \\ \psi_{bu}\left(\sigma_{xz}, \sigma_{xx}\right) & = C\left[\frac{\left|\sigma_{xz}\left(1-\alpha\delta\partial_x b^2\right) - 2\alpha\delta\sigma_{xx}\partial_x b\right|^{p-1}}{\left(1+\alpha\delta\partial_x b^2\right)^{p+\frac{1}{2}}}\left(\sigma_{xz}\left(1-\alpha\delta\partial_x b^2\right) - 2\alpha\delta\sigma_{xx}\partial_x b\right)\right]_{|z=b} \\ \psi_{u}\left(\sigma_{xz}, \sigma_{xx}, u_{|z=b}, w\right) & = u_{|z=b} + \int_b^z\left(\left\|\sigma_e\right\|^{p-1}\sigma_{xz}\right)\mathrm{d}z \\ & \quad -\alpha\delta\left(\partial_x\left(\int_b^z w\mathrm{d}z\right) + w_{|z=b}\partial_x b\right) \\ \psi_{w}\left(u_{|z=b}, u\right) & = u_{|z=b}\partial_x b - \int_b^z \partial_x u\mathrm{d}z \end{array}\right. \tag{33}$$

with

$$\left\|\sigma_e\right\| = \sqrt{\sigma_{xz}^2 + \alpha\delta\sigma_{xx}^2} \tag{34}$$

It is useful to develop functions ψ_{xx}, ψ_{bu} and ψ_u in the form of Taylor series. Assuming that the quantities σ_{xz}, σ_{xx}, $\partial_x u$ and w may be written as follows:

$$\varphi = \varphi^{(1)} + O\left(\left(\beta+\alpha\delta\right)^2\right)$$

with

$$\varphi^{(1)} = \varphi^{(0)} + \beta\varphi^{(1,\beta)} + \alpha\delta\varphi^{(1,\alpha\delta)}$$

Then one can expand ψ_{xx}, ψ_{bu} and $\psi_u$at order 1 for Regimes A and B, at order 2 for Regime C:

$$\begin{cases} \psi_{xx}\left(\sigma_{xz},\sigma_{xx},u\right) & = 2\left|\sigma_{xz}^{(0)}\right|^{1-p}\left(\left(\partial_x u^{(0)}+\beta\partial_x u^{(1,\beta)}+\alpha\delta\partial_x u^{(1,\alpha\delta)}\right)\right. \\ & \left. +\left(1-p\right)\frac{\partial_x u^{(0)}}{\sigma_{xz}^{(0)}}\left(\beta\sigma_{xz}^{(1,\beta)}+\alpha\delta\sigma_{xz}^{(1,\alpha\delta)}+\frac{1}{2}\alpha\delta\frac{\left(\sigma_{xx}^{(0)}\right)^2}{\sigma_{xz}^{(0)}}\right)\right)+O\left(f\left(\alpha,\beta,\delta\right)\right) \\ \psi_{bu}\left(\sigma_{xz},\sigma_{xx}\right) & = C\left(\left|\sigma_{xz|z=b}^{(0)}\right|\right)^{p-1}\left(\sigma_{xz|z=b}^{(0)}+p\left(\beta\sigma_{xz|z=b}^{(1,\beta)}+\alpha\delta\sigma_{xz|z=b}^{(1,\alpha\delta)}\right)\right. \\ & \left. -\alpha\delta\left(2p\partial_x b\sigma_{xx|z=b}^{(0)}+\sigma_{xz|z=b}^{(0)}\left(\partial_x b\right)^2\left(2p+\frac{1}{2}\right)\right)\right)+O\left(f\left(\alpha,\beta,\delta\right)\right) \\ \psi_u\left(\sigma_{xz},\sigma_{xx},u_{|z=b},w\right) & = u_{|z=b}+\int_b^z\left|\sigma_{xz}^{(0)}\right|^{p-1}\left(\sigma_{xz}^{(0)}+p\left(\beta\sigma_{xz}^{(1,\beta)}+\alpha\delta\sigma_{xz}^{(1,\alpha\delta)}\right)+\alpha\delta\frac{(p-1)}{2}\frac{\left(\sigma_{xx}^{(0)}\right)^2}{\sigma_{xz}^{(0)}}\right)\mathrm{d}z \\ & -\alpha\delta\left(\partial_x\left(\int_b^z w^{(0)}\mathrm{d}z\right)+w_{|z=b}^{(0)}\partial_x b\right)+O\left(f\left(\alpha,\beta,\delta\right)\right) \end{cases}$$

where the residual function f depends on α and δ as follows:

- if α is small and δ is $O(1)$ (Regime A), $f(\alpha,\beta,\delta)=(\beta+\alpha)^2$
- if δ is small and α is $O(1)$ (Regime B), $f(\alpha,\beta,\delta)=(\beta+\delta)^2$
- if α and δ are small (Regime C), $f(\alpha,\beta,\delta)=(\beta+\alpha\delta)^2$

We define the operator:

$$\Psi_h : X=\left(\sigma_{xz},\sigma_{xx},\pi,u_{|z=b},u,w\right)\mapsto\Psi_h(X)=\left(\psi_{xz},\psi_{xx},\psi_\pi,\psi_{bu},\psi_u,\psi_w\right) \tag{35}$$

where $X=\left(\sigma_{xz},\sigma_{xx},\pi,u_{|z=0},u,w\right)$ and $\psi_\square$ are defined by equations (33). Then, X is solution of (20) (23) and(24) if $X=\Psi_h(X)$ (X is a fixed point of Ψ_h).

4.2. The averaged mass equation

We define the flow rate q and the mean velocity as follows:

$$q=\int_b^H u\mathrm{d}z \text{ and } \bar{u}=\frac{1}{h}\int_b^H u\mathrm{d}z \tag{36}$$

hence: $q=h\bar{u}$.

We assume that b is independent of time. We integrate the mass equation over the height, we use the Leibniz integration rule and the free surface equation (22), then we obtain the averaged mass equation:

$$\boxed{\partial_t h+\partial_x q=a \text{ for } (x,t) \text{ in } I\times]0,T[} \tag{37}$$

If the averaged mass equation is satisfied, then the free surface equation (22) is satisfied too.

4.3. Iterates and asymptotic expansions

The asymptotic expansions are derived using the following general iterative algorithm $(\mathcal{A})$.

$(\mathcal{A})$:

- Let $(X^{(0)},h^{(0)})$ (and potentially $*^{(-1)}$) be the reference flow solution (steady-state uniform flow). Recall that the latter depends on the flow regime considered, see the previous section.
- $k\to(k+1)$, calculate: $X^{(k+1)}=\Psi_h\left(X^{(k)}\right)$, with h given and Ψ_h defined by (35).

One need to iterate once (respectively twice i.e. $k=1$) in order to obtain 1st order (respectively 2nd order) asymptotic solutions.

This iterative way to calculate the exact solutions and the asymptotic models remains a classical perturbation method. In the present study, it has to be adapted in view to derive unified expressions, valid for all regimes. As a matter of fact, the right expansion terms for each regime have to be kept. Below, the expansion algorithms have to be adapted following the regime considered.

Recall that in [33], it is proved, in the newtonian case, with adherence on a flat bottom, that the present algorithm leads to the Saint-Venant's equations. In other words at $k = 1$ the first order asymptotical equations with respect to ε are mathematically proved.

As mentioned above, the algorithm definition and the source terms $\Psi_h\left(X^{(k)}\right)$ depend on the regime considered.

Regime A: α and β small, $\delta = O\left(1\right)$.
The iterative scheme to compute $(\sigma_{xx}^{(k+1)}, \sigma_{xz}^{(k+1)}, u_{|z=b}^{(k+1)}, u^{(k+1)}, w^{(k+1)}, \pi^{(k+1)})$ from the previous iterate value writes:

$$\left\{\begin{array}{ll} \pi^{(k+1)} & = \psi_\pi\left(\sigma_{xz}^{(k)}, \sigma_{xx}^{(k)}, u^{(k-1)}, w^{(k-1)}\right) \\ \sigma_{xz}^{(k+1)} & = \psi_{xz}\left(\sigma_{xx}^{(k)}, \pi^{(k+1)}, u^{(k)}, w^{(k)}\right) \\ u_{|z=b}^{(k+1)} & = \psi_{bu}\left(\sigma_{xz}^{(k+1)}, \sigma_{xx}^{(k)}\right) \\ u^{(k+1)} & = \psi_u\left(\sigma_{xz}^{(k+1)}, \sigma_{xx}^{(k)}, u_{|z=b}^{(k+1)}, w^{(k)}\right) \\ \sigma_{xx}^{(k+1)} & = \psi_{xx}\left(\sigma_{xz}^{(k+1)}, \sigma_{xx}^{(k)}, u^{(k+1)}\right) \\ w^{(k+1)} & = \psi_w\left(u_{|z=b}^{(k+1)}, u^{(k+1)}\right) \end{array}\right. \tag{38}$$

Regime B: β, δ small, $\alpha = O\left(1\right)$. (Since $\varepsilon^2 = \alpha\delta$ then $\varepsilon^2 = O\left(\delta\right)$).
The iterative scheme writes:

$$\left\{\begin{array}{ll} \sigma_{xz}^{(k+1)} & = \psi_{xz}\left(\sigma_{xx}^{(k)}, \pi^{(k)}, u^{(k)}, w^{(k)}\right) \\ u_{|z=b}^{(k+1)} & = \psi_{bu}\left(\sigma_{xz}^{(k+1)}, \sigma_{xx}^{(k)}\right) \\ u^{(k+1)} & = \psi_u\left(\sigma_{xz}^{(k+1)}, \sigma_{xx}^{(k)}, u_{|z=b}^{(k+1)}, w^{(k)}\right) \\ \sigma_{xx}^{(k+1)} & = \psi_{xx}\left(\sigma_{xz}^{(k+1)}, \sigma_{xx}^{(k)}, u^{(k+1)}\right) \\ \pi^{(k+1)} & = \psi_\pi\left(\sigma_{xz}^{(k+1)}, \sigma_{xx}^{(k+1)}, u^{(k)}, w^{(k)}\right) \\ w^{(k+1)} & = \psi_w\left(u_{|z=b}^{(k+1)}, u^{(k+1)}\right) \end{array}\right. \tag{39}$$

Recall that in both cases (Regime A and B), the algorithm is iterated once ($k = 0$) or twice ($k = 1$) depending on the asymptotic order sough.

As mentioned in AppendixA, all the calculations and "source terms" $\psi_\Box$ are unchanged if using the Prandtl shift at 0th order and at order 1 *while considering 1st order terms in the geometrical ratio* ε too (i.e. skipping terms in $\alpha\delta$, $\alpha\delta = \varepsilon^2$). Moreover the fixed point like formulation (35) and the iterative schemes (38)-(39) still apply. Extra terms appear in the pressure expression at order 1 in the case $\alpha = 0(1)$ (Regime B) but this expression is useless to derive the present asymptotical models. We refer the reader to AppendixA.

5. Error estimates

Let $(X^{(0)}, h^{(0)})$ be the uniform stationary solution at order 0 i.e. constant functions in (x, t). It is the reference flow defined in Section 3.

We set:

$$\begin{array}{ll} X^{(0)} & = \left(\sigma_{xz}, \sigma_{xx}, \pi, u_{|z=b}, u, w\right)^{(0)} \\ & = \left(\lambda\left(H-z\right), 0, H-z, C\lambda^p H^p, \left[C\lambda^p H^p + \frac{\lambda^p}{p+1}\left(H^{p+1} - \left(H-z\right)^{p+1}\right)\right], 0\right) \end{array}$$

We set the error function:

$$\varepsilon^{(k)}\left(\varphi\right) = \left[\varphi^{ex} - \varphi^{(k)}\right]$$

where φ^{ex} denotes the exact solution. We consider below the case $k=1$, next the case $k=2$.

5.1. *Formal error estimates*

We assume that if $\varepsilon^{(k)}(\varphi) = O\left(s\right)$, s being a small parameter, then $\varepsilon^{(k)}\left(\partial_p \varphi\right) = O\left(s\right), \ \forall p \geq 0$.

First we consider **Regime A** for which α and β are small, δ is $O\left(1\right)$. At first iterate of algorithm $(\mathcal{A})$ we obtain error estimates for the pressure, the velocity field and stress tensor:

$$\begin{array}{ll} \varepsilon^{(1)}\left(\pi\right) & = O\left(\alpha\right) \\ \varepsilon^{(1)}\left(\sigma_{xz}\right) & = O\left(\alpha\delta + \beta\right) = O(\alpha+\beta) \text{ since } \delta = O(1). \\ \varepsilon^{(1)}\left(u_{|z=b}\right) & = O\left(C\left(\alpha\delta+\beta\right)\right) = O(C(\alpha+\beta)) \text{ since } \delta = O(1). \\ \varepsilon^{(1)}\left(u\right) & = O\left(\alpha + \varepsilon^{(1)}\left(\sigma_{xz}\right) + \varepsilon^{(1)}\left(u_{|z=b}\right)\right) = O\left(\left(1+C\right)\left(\alpha+\beta\right)\right) \\ \varepsilon^{(1)}\left(\sigma_{xx}\right) & = O\left(\alpha + \varepsilon^{(1)}\left(u\right) + \varepsilon^{(1)}\left(\sigma_{xz}\right)\right) = O\left(\left(1+C\right)\left(\alpha+\beta\right)\right) \\ \varepsilon^{(1)}\left(w\right) & = \varepsilon^{(1)}\left(u\right) = O\left(\left(1+C\right)\left(\alpha+\beta\right)\right) \end{array} \tag{40}$$

where we expanded nonlinearities of the norm of the stress tensor in the error function with respect to $\varepsilon^{(k)}\left(\varphi\right)$ using the fact that $\varphi^{ex} = \varphi^{(k)} + \varepsilon^{(k)}\left(\varphi\right)$. Thus, after first iterate ($k=1$), we obtain the zero-th order solution with respect to (α, β).

After iterate (2) of Algorithm $(\mathcal{A})$ we get the following error functions:

$$\begin{array}{ll} \varepsilon^{(2)}\left(\pi\right) & = O\left(\alpha\left(\varepsilon^{(1)}\left(\sigma_{xz}\right) + \varepsilon^{(1)}\left(\sigma_{xx}\right) + \beta\right)\right) \\ \varepsilon^{(2)}\left(\sigma_{xz}\right) & = O\left(\alpha\left(\varepsilon^{(1)}\left(\sigma_{xz}\right) + \varepsilon^{(1)}\left(\sigma_{xx}\right) + \beta\right) + \beta\left(\varepsilon^{(1)}\left(u\right) + \varepsilon^{(1)}\left(w\right)\right)\right) \\ \varepsilon^{(2)}\left(u_{|z=b}\right) & = O\left(C\left(\alpha\left(\varepsilon^{(1)}\left(\sigma_{xz}\right) + \varepsilon^{(1)}\left(\sigma_{xx}\right) + \alpha + \beta\right) + \beta\left(\varepsilon^{(1)}\left(u\right) + \varepsilon^{(1)}\left(w\right)\right)\right)\right) \end{array} \tag{41}$$

$$\begin{array}{ll} \varepsilon^{(2)}\left(u\right) & = \varepsilon^{(2)}\left(\sigma_{xx}\right) = \varepsilon^{(2)}\left(w\right) \\ & = O\left(\left(1+C\right)\left(\alpha\left(\varepsilon^{(1)}\left(\sigma_{xz}\right) + \varepsilon^{(1)}\left(\sigma_{xx}\right) + \alpha + \beta\right) + \beta\left(\varepsilon^{(1)}\left(u\right) + \varepsilon^{(1)}\left(w\right)\right)\right) + \alpha\varepsilon^{(1)}\left(w\right)\right) \end{array} \tag{42}$$

Again we expanded nonlinearities of the square of the velocity, and the stress tensor norm in the error function with respect to $\varepsilon^{(k)}\left(\varphi\right)$. After second iterate ($k=2$), we obtain the first order solution with respect to (α, β).

Using the same approach we get similar results for **Regime B**, where δ and β are small, α is $O\left(1\right)$.

Regime C (α, δ and β small) is contained in both Regime A and Regime B.

Finally, after iterate (2) of Algorithm $(\mathcal{A})$, the solution obtained $\left(X^{(2)}, h^{(2)}\right)$ is an asymptotic solution of the free surface Navier-Stokes equations at order 1:

- in case of Regime A, it is first order with respect to (α, β);
- in case of Regime B it is first order with respect to (β, δ);
- in case of Regime C it is first order with respect to (α, β, δ).

5.2. Relationship between the slip coefficient C, the global error and the regime

Let us consider Regime A (α, β small). If the slip coefficient C is assumed to be $O(1)$ at most, then the coefficient $C(\alpha + \beta)$ is a small parameter and Algorithm($\mathcal{A}$) gives order 0 asymptotic solutions for $k = 1$ and order 1 solutions for $k = 2$.

In the case of Regime B (δ, β small), let us set: $S = \max(C\delta, C\beta)$. If S is small (i.e. $C \ll \max(1/\delta, 1/\beta)$) then, as mentioned above one can iterate Algorithm($\mathcal{A}$) to obtain order 1 or order 2 asymptotic solutions.

If S is $O(1)$ or larger then the a-priori friction model at bottom is the a-priori dominating error term (see Eqn (41)-(42) by replacing α by δ).
Hence, following the error estimates in the case Regime B, *the global error on the solution is dominated by the error on the velocity at bottom.*
Nevertheless, the present iterative process gives the right asymptotic solution since the mean slope must vanish for large slip coefficient following the relationship (29). And, in case of a vanishing mean slope, the reference solution writes:

$$\left(\sigma_{xz}, \sigma_{xx}, \pi, u_{|z=b}, u, w\right)^{(0)} = (0, 0, (H - z), C\lambda^p H^p, C\lambda^p H^p, 0)$$

Next, we obtain:

$$\psi_{bu}(\sigma_{xz}, \sigma_{xx}) = \left(1 - \frac{\lambda^p}{p+1}\right) H^p + O\left(\delta\left(1 + \frac{\lambda^p}{p+1}\right) + \beta\right)$$

Next, the following error estimate can be stated similarly as before:

$$\begin{aligned} \varepsilon^{(1)}\left(u_{|z=b}\right) &= \psi_{bu}(\sigma_{xz}, \sigma_{xx}) - \psi_{bu}\left(\sigma_{xz}^{(1)}, 0\right) \\ &= O(\delta + \beta) \end{aligned}$$

Hence the resulting velocity field at bottom u_b is order 1 with respect to small parameters; and the present iterative process give asymptotical solutions at the orders required, in both regimes.

6. Expressions of the 0th order exact solutions

One of the forthcoming goal is to derive one-equation models in h variable only, by applying Algorithm ($\mathcal{A}$). Then it will give asymptotic models for regimes A, B, C, and allow to derive an unique equation valid for these three regimes and with the different boundary conditions at bottom.

As a first step, we need to derive the exact expression of the unknown at order 0.

For a sake of clarity, the solutions are written using the mean slope coordinate system. Recall the latter and the Prandtl's coordinate system lead to the same expressions for the terms we need here, see AppendixA.

6.1. Case of Regime A: α small

We have: α, β small and $\delta = O(1)$; also since $\varepsilon^2 = \alpha\delta$ then $\varepsilon^2 = O(\alpha)$. Using the formulation (38), at zero-th order the pressure is the hydrostatic one:

$$\pi^{(0)} = H - z \tag{43}$$

The shear stress σ_{xz} is linear in z:

$$\sigma_{xz}^{(0)} = \lambda(H - z) - \delta\partial_x\left(\int_z^H \pi \mathrm{d}z\right) = (H - z)(\lambda - \delta\partial_x H) \tag{44}$$

To compute the velocity profile we need the modulus of the deviatoric tensor:

$$\left\|\sigma^{(0)}\right\| = \left|\sigma_{xz}^{(0)}\right| = (H - z)\left|\lambda - \delta\partial_x H\right|$$

Remark 16. The absolute value present in the modulus of the deviatoric tensor above can not be removed. As a matter of fact, for a vanishing mean slope λ, the sign of σ_{xz} is the same as the sign of the local slope $\partial_x b$.

The streamwise velocity component writes:

$$u^{(0)} = (\lambda - \delta\partial_x H)\,|\lambda - \delta\partial_x H|^{p-1}\left(Ch^p + \frac{1}{p+1}\left(h^{p+1} - (H-z)^{p+1}\right)\right) \tag{45}$$

Then the discharge writes:

$$q_A^{(0)} = \int_b^H u^{(0)}\mathrm{d}z = (\lambda - \delta\partial_x H)\,|\lambda - \delta\partial_x H|^{p-1}\,h^{p+1}\left(C + \frac{h}{p+2}\right) \tag{46}$$

6.2. Case of Regime B: δ small

We have: β, δ small and $\alpha = O(1)$; then $\varepsilon^2 = O(\delta)$. We use the iterative scheme (39). Note that unlike Regime A, the velocity has to be calculated before the pressure.

The shear stress is still linear in z, but as we don't have the corrective pressure term $\delta\partial_x H$, $\sigma_{xz}^{(0)}$, it is always positive:

$$\sigma_{xz}^{(0)} = \lambda\,(H-z)$$

We easily calculate the velocity and the discharge rate:

$$u^{(0)} = \lambda^p\left(Ch^p + \frac{1}{p+1}\left(h^{p+1} - (H-z)^{p+1}\right)\right)$$

$$q_B^{(0)} = \lambda^p h^{p+1}\left(C + \frac{h}{p+2}\right) \tag{47}$$

Given the velocity, we obtain the pressure:

$$\pi^{(0)} = (H-z)\left(1 + \lambda\alpha\left(\partial_x H - 2\frac{h^{p-1}\,(pC+h)}{(H-z)^p}\partial_x h\right)\right) \tag{48}$$

Remark 17. The pressure expression (48) is singular at the free surface, hence the normal stress components $(\sigma_{xx}, \sigma_{zz})$ too. It is the well-known singularity at the free-surface mentioned for example in [28, 37, 38]. This singularity, already described differently in [38], appears here at order 0 since it is the multi-regime expression. In the sequel, the singularity does not prevent to pursue the calculations (in the two-equations model in particular) since we consider order 0 in regime and order 1 in the geometrical ratio ε (recall $\varepsilon^2 = \alpha\delta$).

6.3. Unified expressions

The field expressions above invite us to define the following new variable:

$$\boxed{\Lambda = \begin{cases} \lambda - \delta\partial_x H & \text{in the case of Regime A} \\ \lambda & \text{in the case of Regime B and C} \end{cases}} \tag{49}$$

Note that: $\lambda - \delta\partial_x H = \gamma^{-1}[\tan(\theta) - \varepsilon\partial_x H]$.
Using this variable definition Λ, the shear stress at bottom $\sigma_{xz}^{(0)}(z=b)$, the longitudinal velocity $u^{(0)}(z)$ and the discharge $q^{(0)}$ write in an unified way as follows:

$$\boxed{\left\{\begin{array}{rcl} \sigma_{xz}^{(0)}|_b &=& \Lambda h \\ u^{(0)}(z) &=& \Lambda\,|\Lambda|^{p-1}\left(Ch^p + \frac{1}{p+1}\left(h^{p+1} - (H-z)^{p+1}\right)\right) \\ q^{(0)} &=& \Lambda\,|\Lambda|^{p-1}\,h^{p+1}\left(C + \frac{h}{p+2}\right) \end{array}\right.} \tag{50}$$

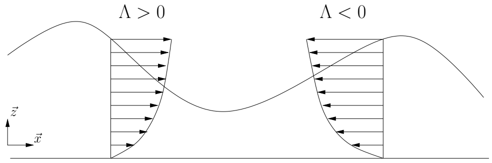


Figure 4: At order 0, Regime A, the sign of Λ (local value) indicates the local direction of the flow (equation (50)).

For each field φ above, the expression corresponding to a given regime is recovered by setting:

$$\varphi_B^{(0)} = \varphi_{(\Lambda=\lambda)}^{(0)} \text{ and } \varphi_A^{(0)} = \varphi_{(\Lambda=\lambda-\delta\partial_x H)}^{(0)}$$

Also, we have:

$$\begin{cases} u^{(0)}|_b &= C\Lambda\,|\Lambda|^{p-1}\,h^p \\ u^{(0)}|_H &= \Lambda\,|\Lambda|^{p-1}\,h^p(C+\frac{1}{p+1}h) \end{cases}$$

Remark 18. In its dimensional form (case Regime A), Λ writes:

$$\Lambda = \frac{\varrho g H^*\left(\sin\theta - \cos\theta\partial_x H\right)}{\frac{\mu_e U^*}{H^*}}$$

It is the ratio between a term related to the gravity and the scaling of the deviatoric stress tensor σ. Indeed the gravity term is composed of the streamwise gravity component $\varrho g H^* \sin\theta$ and the normal component of the gravity scaled by the slope of the free surface $(-\varrho g H^* \cos\theta\partial_x H)$. As a result, the quantity Λ represents the gravity contribution due to the local slope of the free-surface.

In Regime B, the velocity u depends on the mean-slope contribution only, while in Regime A the velocity u depends on the local slope of the free-surface too.

In Regime A, if Λ is negative, then $q^{(0)}$ is negative, see figure 4.

6.4. *Case of Regime C: α and δ small*

Regime C is a special case of both Regime A and Regime B. It is included in Regime A since α is small, and in Regime B since δ is small. As a consequence one recover Regime C flow values (stress, pressure, velocity and flow rate) by setting $\delta = 0$ in Regime A or $\alpha = 0$ in Regime B. Inasmuch as α does not appear in the flow rate of Regime B, the flow rate of Regime C derives from the Regime B one. Then:

$$q_C^{(0)} = q_B^{(0)}$$

7. First order solutions

Next iteration of algorithms (38) and (39) is computed using Sage software ([40]). It gives the explicit expressions of all fields. Two general forms of the solutions are exposed while we present the exact expression of the discharge in the simplest form .

7.1. *General form of the solutions*

Formally, first order solutions can be written in a general form as:

$$\varphi = \quad \varphi^{(1)} + \mathcal{O}\left(\left(\sum_{\phi\in\Phi}\phi\right)^2\right) = \varphi^{(0)} + \sum_{\phi\in\Phi}\phi\varphi^{(1,\phi)} + \mathcal{O}\left(\left(\sum_{\phi\in\Phi}\phi\right)^2\right) \tag{51}$$

with

$$\Phi = \begin{cases} \{\alpha,\beta\} & \text{(regime A)} \\ \{\beta,\delta\} & \text{(regime B)} \\ \{\alpha,\beta,\delta\} & \text{(regime C)} \end{cases}$$

In the present context, first order solutions would mean first order with respect to each (potentially) small parameters (β,α,δ). Then, any first order solution may write as follows:

$$\varphi^{(1)} = \varphi^{(0)} + \beta\varphi^{(1,\beta)} + \beta\delta\varphi^{(1,\beta\delta)} + \mathcal{O}((\sum_{\phi\in\Phi}\phi)^2 + \varepsilon^2) \tag{52}$$

Nevertheless, first order solutions in addition with respect to the geometrical ratio ε would be already an original and rich contribution to the derivations. Thus, in all the sequel, we call *first order solutions*, solutions order one with respect to the regime considered *and* order one with respect to the geometrical parameter ε. In others words, all terms in $\alpha\delta = \varepsilon^2$ are omitted. Then, it will appear that the solutions can be written in the following form:

$$\begin{aligned} \varphi^{(1)} \;\; = \;\; & \varphi^{(0)} + \beta A_1^{\varphi}(h,\Lambda)\partial_x h + \beta A_2^{\varphi}(h,\Lambda)\partial_x\Lambda + \beta A_3^{\varphi}(h,\Lambda)\partial_t\Lambda \\ & + \beta A_4^{\varphi}(h,\Lambda)\partial_x b + \beta A_5^{\varphi}(h,\Lambda)a + \mathcal{O}((\beta+\alpha+\delta)^2+\varepsilon^2) \end{aligned} \tag{53}$$

with the terms A_k^{φ} depending on h and Λ, $\Lambda \equiv \Lambda(\delta,\partial_x H)$ defined by (49) hence:

$$\partial_x\Lambda = -\delta\partial_{xx}^2 H \text{ and } \partial_t\Lambda = -\delta\partial_{xt}^2 H = \delta\partial_{xx}^2 q \tag{54}$$

Let us notice that $\partial_x\Lambda = O(\delta) = \partial_t\Lambda$. Also, the topography variation $\partial_x b$ appears a-priori in all terms $A_\square^{\varphi}$ since the definition of Λ.

7.2. The discharge at order 1

Following the structure above, the discharge order 1 writes:

$$\boxed{\begin{aligned} q^{(1)} \;\; = \;\; & q^{(0)} + \beta A_1^{q}(h,\Lambda)\partial_x h + \beta A_2^{q}(h,\Lambda)\partial_x\Lambda + \beta A_3^{q}(h,\Lambda)\partial_t\Lambda \\ & + \beta A_4^{q}(h,\Lambda)\partial_x b + \beta A_5^{q}(h,\Lambda)a + \mathcal{O}((\beta+\alpha+\delta)^2+\varepsilon^2) \end{aligned}} \tag{55}$$

where the terms $A_\square^{q}(h,\Lambda)$ are presented in AppendixB.

The expression above is unified in the sense that it is valid for all regime by setting $\Lambda = \lambda$ in the case of regime B and C.

Remark 19. As mentioned above, in Regime A (α,β small), the discharge at first order could be developed in the following form:

$$q^{(1)} = q^{(0)} + \beta q^{(1,\beta)} + \beta\delta q^{(1,\beta\delta)} + \mathcal{O}\left((\alpha+\beta)^2 + \varepsilon^2\right) \tag{56}$$

where $q^{(0)}$ is the zero-th order term defined by (50). In the case of Regime B (δ,β small) and Regime C (the three numbers are small), it simplifies as follows:

$$q^{(1)} = q_B^{(0)} + \delta q^{(1,\delta)} + \beta q^{(1,\beta)} + \mathcal{O}\left((\delta+\beta)^2 + \varepsilon^2\right) \tag{57}$$

where $q_B^{(0)}$ is the zero-th order term defined by (47) and:

$$q^{(1,\delta)} = -p\lambda^{p-1}h^{p+1}\left(C + \frac{h}{p+2}\right)\partial_x h$$

Remark 20. If one set $\delta = 1$ and $\lambda = 0$ in Equation (20), we obtain all fields calculated in [28] (which have been fixed by considering the scaling a-priori $\sigma_{xz}^* = \varepsilon\varrho g h$, see [28] page 238).

8. One-equation models

In this section, first, we deduce classically from the average mass equation and the exact solution expressions above, the one-equation models at order 0 and at order 1. These models remain consistent for the regimes A, B, C, and for any boundary condition at bottom (from adherence to pure-slip through a non-linear friction law), with the limitation that the mean slope dimensionless number λ and the slip coefficient C of the reference flow respect the relationship (28).
In the 0th order case, we give the expression of the steady-state model both for Regime A and Regime B. Next, we derive the equation obtained if pure slip (vanishing friction) is imposed at bottom; also we detail the model obtained in the newtonian case ($p = 1$).

Let us point out that all derivations of the equations are similar for 3D flows. On the contrary, the discussion related to the steady state solution is specific to the 2D (vertical) case.

8.1. The 0th order unified one-equation model

The fluid height verifies the averaged mass equation (37) which leads to the steady height equation by replacing the discharge rate by its value. Then we obtain the zero-th order unified model for both regimes A and B:

$$\boxed{\partial_t h + \partial_x \left(\Lambda \left|\Lambda\right|^{p-1} h^{p+1} \left(C + \frac{h}{p+2} \right) \right) = a} \tag{58}$$

where Λ is defined by (49).

> Recall the coefficient C denotes the slip coefficient which can vary from 0 (adherence at bottom) to infinity (pure slip case, see below the resulting equations).

Once the solution of the unified model (58) is computed, the velocity profile, the pressure and the stress tensor are given by the formal expressions.

Remark 21. The 0th order unified model is the same for regimes B and C. Therefore the fluid height solution is identical. Nevertheless the pressure and the normal stress are different since they include terms in α.

8.2. Steady-state solution

Let us calculate the steady solution order 0 in both regimes A and B. Since the flow is steady, we consider only the case of vanishing source term ($a = 0$). A direct integration gives:

$$\Lambda \left|\Lambda\right|^{p-1} h^{p+1} \left(C + \frac{h}{p+2} \right) = q_0 \tag{59}$$

with q_0 the initial discharge rate. Let us develop (59) for both regimes. The steady-state equation order 0 for both regimes writes:

$$\boxed{\partial_x H = \partial_x (h + b) = \frac{1}{\delta} \left(\lambda - \frac{\operatorname{sgn}(q_0)}{h} \left(\frac{|q_0|}{h \left(C + \frac{h}{p+2} \right)} \right)^{1/p} \right)} \tag{60}$$

Using the expressions of δ and λ in function of γ, see (16), we have:

$$\partial_x h + \partial_x b = \frac{1}{\varepsilon} \left(\tan\theta - \gamma \frac{\operatorname{sgn}(q_0)}{h} \left(\frac{|q_0|}{h \left(C + \frac{h}{p+2} \right)} \right)^{1/p} \right)$$

We remark that in Regime A and if $\lambda = 0$ (vanishing mean slope) then at order 0 the steady solution satisfies: $\partial_x H < 0$ for $q_0 > 0$.

In Regime B (recall δ small and $\partial_x H \leq O\left(\frac{1}{\delta}\right)$), it follows from (59) that h is the solution of the following equation:

$$\lambda^p h^{p+1}\left(C + \frac{h}{p+2}\right) = q_0$$

In other words, the steady-state solution Regime B is such that the depth h is constant. That is to say the surface is a translation of the bottom shape. Indeed we recover the reference flow as defined by equation (28).

8.3. *The* 1*st order unified one-equation model*

At first order, Equation (37) writes:

$$\boxed{\partial_t h + \partial_x\left(q^{(0)} + \beta\left(A_1^q(h,\Lambda)\partial_x h + A_2^q(h,\Lambda)\partial_x\Lambda + A_3^q(h,\Lambda)\partial_t\Lambda + A_4^q(h,\Lambda)\partial_x b + A_5^q(h,\Lambda)a\right)\right) = a} \quad (61)$$

with $q^{(0)}$ defined by (50), the coefficients A_k^q are given in AppendixB, and the terms $\partial_\square\Lambda$ are given in (54).

For Regimes B and C the unified model (61) can be written more easily as follows:

$$\partial_t h + \partial_x\left(\lambda^{p-1}(\lambda - p\delta\partial_x H)h^{p+1}\left(C + \frac{h}{p+2}\right) + \beta A_1^q(h, \Lambda = \lambda)\right) = a$$

Remark 22. For low Reynolds flows β is negligible (eg. ice flows), then the unified model order 1 (61) equals the zero-th order model (58). In other words, in this case, Equation (58) is order 1.

8.4. *Pure slip case*

Let us recall that it is not worth to focus on the 1st order model with large values of the slip coefficients C since the global error on the solution is dominated by the error on the velocity at bottom (see Subsection 5.2).

Let us consider the 0th order unified model in the pure slip case i.e. $C \to \infty$. Then Equation (58) writes:

$$\frac{1}{C}\partial_t h + \partial_x\left(\Lambda\left|\Lambda\right|^{p-1} h^{p+1} + \frac{1}{C}\frac{1}{p+2}\Lambda\left|\Lambda\right|^{p-1} h^{p+2}\right) = \frac{a}{C}$$

Passing to the limit in C, one obtains the so-called plug model (see Remark 13):

$$\partial_x\left(\Lambda\left|\Lambda\right|^{p-1} h^{p+1}\right) = 0 \quad (62)$$

Note that it is a steady state solution and it depends only on the sliding-friction power law.

In Regime B, $\Lambda = \lambda$ is constant hence: $\partial_x\left(h^{p+1}\right) = 0$. Thus, in the pure slip case Regime B, the solution is a constant height. In other words, the surface shape is a simple translation of the bottom shape.

In Regime A, the equation (62) writes: $\Lambda = \operatorname{sgn}(q_0)\,\frac{1}{h}\left|\frac{q_0}{h}\right|^{1/p}$, with q_0 the steady uniform discharge. Since $\Lambda = (\lambda - \delta\partial_x H)$, Eqn (58) writes into an ordinary differential equation:

$$\partial_x h = \frac{1}{\delta}\left(\lambda - \operatorname{sgn}(q_0)\,\frac{1}{h}\left|\frac{q_0}{h}\right|^{1/p}\right) - \partial_x b \quad (63)$$

It is the one-equation model Regime A if pure slip at bottom

8.5. *Newtonian rheology case*

Let us detail the newtonian case ($p = 1$) at order 0. In the case of Regime B, with the mean velocity as referent velocity, we obtain the following non-linear advection equation:

$$\partial_t h + \lambda \partial_x \left(C h^2 + \frac{h^3}{3} \right) = a$$

with the velocity value equal to $\lambda h \,(2C + h)$. At order 0 and in the case of Regime A, we obtain the advective term of Regime B plus an extra diffusive term. The model reads:

$$\partial_t h + \partial_x \left(\left(\lambda - \delta \partial_x \left(h + b\right)\right) \left(C h^2 + \frac{h^3}{3} \right) \right) = a$$

At order 1, the equation coefficients are much longer and are obtained by setting $p = 1$ in (61).

9. Preliminary two-equation model and gradually varied flow model

In this section, we seek to derive two-equation models in variable (h, q) (Saint-Venant or shallow-water like models if using the hydraulic terminology). Starting from the depth-integrated mass and momentum equations, all terms have to be developed using order 0 or 1 field expressions to close the two-equation system (the order required depending on the multiplicative factors). In this section, as a first step, a preliminary two-equations model is derived. The model is "preliminary" since it is not fully consistent with the free-surface Navier-Stokes system since the basal friction term is developed at order 0 only while order 1 is required (see the discussion below). Nevertheless, from this preliminary two-equation model, we can derive a useful model in practice: a "gradually varied" flow model i.e. modeling a non-uniform flow in which depth changes gradually only. The gradually varied flow model is discussed has it would be in a hydraulic context (concepts of critical-line depth value, sub and super critical values etc). Two-equation models fully consistent with the free-surface primitive system and valid for the multi-regime flows are derived in next section.

Let us recall that a two-equation model implies the one-equation model (same order) but the reciprocal is not true. Then, the advantages of a two-equation model are the following. a) in the case of a steady-state uniform flow, the model simplifies naturally to the balance friction - slope law (the law acting as a recall term in the RHS of the momentum equation). This feature is important since the asymptotics derived are perturbations of reference flows (which are the steady-state uniform ones). b) A potential blow-up of the one-equation model solution in finite time since the linear stability threshold may be overpassed. This result, well-known in the newtonian case see e.g. [6], is circumvented if considering the two-equation model. c) the two-equation model may allow to handle more consistently wet-dry fronts ($h \to 0$) (nevertheless, this feature is out of scope of the present study).

9.1. *Fundamentals of multi-regime shallow-water type system*

We consider the first momentum equation in its primal form, we integrate it over the depth, we use the boundary conditions, and together with the mass conservation equation we obtain:

$$\boxed{\left\{ \begin{array}{l} \partial_t h + \partial_x q = a \\ \partial_t q + \partial_x \left(\displaystyle\int_b^H u^2 \mathrm{d}z \right) - \dfrac{\alpha\delta}{\beta} \partial_x \left(\displaystyle\int_b^H \sigma_{xx} \mathrm{d}z \right) + \dfrac{\delta}{\beta} \partial_x \left(\displaystyle\int_b^H \pi \mathrm{d}z \right) \\ \quad + \dfrac{\delta}{\beta} \left(\pi_{|z=b} + \alpha \sigma_{xx|z=b} \right) \partial_x b = \dfrac{1}{\beta} \left(\lambda h - \sigma_{xz|z=b} \right) \end{array} \right.} \tag{64}$$

In the sequel, for a sake of simplicity we set a (the mass balance source term) to 0; except if the contrary is mentioned.

Remark 23. In the case of steady-state uniform flow with flat bottom, Equation (64) writes: $\lambda h = \sigma_{xz|z=b}$. It means that the fluid elevation is determined by the shear stress at bottom. This equation gives the law elevation - velocity (or discharge) of the steady uniform flow (like it is done by the Manning-Strickler (or Chezy) law in hydraulics). In other words, the friction - slope law acts as a recall term in the RHS of the momentum equation.

Assuming that the tangential velocity u is close to its depth-average value $\bar{u}$, it follows: $\int_b^H u^2 \mathrm{d}z \approx \frac{q^2}{h}$. Hence in Equation (64), the advective term $\int_b^H u^2 \mathrm{d}z$ is written in the form : $[\int_b^H u^2 \mathrm{d}z - \frac{q^2}{h}]$ ="corrective term".

At order 0, we have:

$$\int_b^H u^2 \mathrm{d}z = \frac{q^2}{h} + \frac{\Lambda^{2p} h^{2p+3}}{(2p+3)\,(p+2)^2} \tag{65}$$

with Λ defined by (49).

Remark 24. In the case $C = 0$ (i.e. no slip) , the integral $\int_b^H u^2 \mathrm{d}z$ can be developed as follows:

$$\int_b^H u^2 \mathrm{d}z = C_1 \frac{q^2}{h} + C_2 \frac{\Lambda^{2p} h^{2p+3}}{(2p+3)\,(p+2)^2} + O(\beta)$$

with C_1 arbitrary and C_2 such that:

$$\frac{(q^{(0)})^2}{h} + \frac{\Lambda^{2p} h^{2p+3}}{(2p+3)\,(p+2)^2} = C_1 \frac{(q^{(0)})^2}{h} + C_2 \frac{\Lambda^{2p} h^{2p+3}}{(2p+3)\,(p+2)^2} + O(\beta)$$

In the case $p = 1$, Regime C, it gives $C_2 = \frac{1-C_1}{9}$, see the resulting non linear term in H^5 in (90). The classical value considered in the literature is $C_1 = \frac{6}{5}$, see e.g. [5, 6, 33]. This remark is the key point to make the link between forthcoming different parameter form models, see Remark 28.

Let us write the basal shear stress $\sigma_{xz|b}$ order 0 as a function of the flow rate q. We have: $q^{(0)} = \Lambda h \,|\Lambda h|^{p-1}\, h \left(C + \dfrac{h}{p+2}\right)$ and $\sigma^{(0)}_{xz|z=0} = \Lambda h$.

Hence:

$$q^{(0)} = \sigma^{(0)}_{xz|b} \left|\sigma^{(0)}_{xz|b}\right|^{p-1} h \left(C + \frac{h}{p+2}\right)$$

In other respect, the sign of the shear stress is the sign of the local slope Λ:

$$\mathrm{sgn}\left(\sigma^{(0)}_{xz|b}\right) = \mathrm{sgn}\left(q^{(0)}\right) = \mathrm{sgn}\left(\Lambda h\right) = \mathrm{sgn}\left(\Lambda\right) = \frac{\Lambda}{|\Lambda|}$$

Hence the order 0 basal shear stress expression writes:

$$\sigma^{(0)}_{xz|b} = \mathrm{sgn}(\Lambda) \left(\frac{|q^{(0)}|}{h\left(C + \frac{h}{p+2}\right)}\right)^{\frac{1}{p}} \tag{66}$$

Remark 25. In the case $p \geq 2$, the singularity appearing first in the pressure term at order 0 (term in $O(\alpha)$, see (48)) does not appear in the preliminary two-equation model since the terms in order $\frac{\alpha\delta}{\beta}$ are neglected (thus in particular the term in σ_{xx} in (64)). This singularity enhances a modeling problem due to an unphysical rheology at vanishing shear rate. Few possibilities of regularization of the rheology law are possible. One of them is to consider a "residual" newtonian behavior in the Nye-Glen's law (6), see e.g. [38, 41]. Observe that in this case, it has been proved in [42] that, the dispersive behavior of the linearized system (stability analysis) depends on the regularization.

9.2. A preliminary two-equation model

Each term of the second equation of (64) has to be developed at order 0. This implies that each field has to be considered either at order 0 or at order 1, depending on the multiplicative factor and the regime considered. Nevertheless, as a first step, *we consider the 0-order expression of $\sigma_{xz|b}$ in the RHS* of (64). In other respects, all terms in $\dfrac{\alpha\delta}{\beta}$ are neglected. (Since $\alpha\delta = \varepsilon^2$, the latter simplification is correct if $\beta \approx \varepsilon$ for example). Then the resulting two-equation system writes:

$$\boxed{\begin{cases} \partial_t h + \partial_x q = 0 \\ \partial_t q + \partial_x \left(\dfrac{q^2}{h} + \dfrac{\Lambda^{2p} h^{2p+3}}{(2p+3)(p+2)^2} \right) + \dfrac{\delta}{\beta} h \partial_x H \\ \quad = \dfrac{1}{\beta} \left(\lambda h - \operatorname{sgn}(\Lambda) \left(\dfrac{|q|}{h \left(C + \frac{h}{p+2} \right)} \right)^{\frac{1}{p}} \right) + h.o.t. \end{cases}} \tag{67}$$

where Λ is defined by (49) and $h.o.t.$ denotes the "higher-order terms".

The vertical viscous term leading to the “friction term” (the term in $\frac{1}{\beta}$ in the R.H.S.) has to be developed at order 1 in β in view to obtain an asymptotically consistent model. In other words, the basal shear stress $\sigma_{xz|b}$ should be expanded up to first order in β. In the present section, as a first step we consider the preliminary two-equation model based on the derivation of the basal shear stress $\sigma_{xz|b}$ at order 0 only, hence neglecting the higher order terms $h.o.t.$ in the RHS. The resulting system is a-priori not consistent with the free-surface Navier-Stokes system but it is a first step to derive the gradually varied flow model below.

In the case of Regime C (all three parameters small), a consistent two-equation model has been written by Fernandez-Nieto et al. [19]; while in the multi-regime case, a consistent two-equation model is detailed in next section.

9.3. The gradually varied flow model and hydraulic-like concepts

In this section, following the classical hydraulics concept of "gradually varied" and the resulting definitions of hydraulic Froude number, critical-depth line value etc, we present a short analysis of the gradually varied flow model deriving from (67). Let us recall that this model is not fully consistent with the free-surface Navier-Stokes equations; nevertheless it already allows to describe some features of the multi-regime model solutions. A hydraulic-like analysis illustrates the depth behavior in function of (α, β, δ) and C, and depending if a subcritical or a supercritical flow occurs.

For sake of simplicity, we consider here a fluid flowing from left to right on an inclined plane, i.e. $b = 0$, $H = h$ and $\Lambda > 0$, and $a = 0$. Let us investigate the steady flow ($\partial_t h = 0$). The mass conservation implies that the flow rate is constant. The system of equations (67) reduces to the following scalar equation:

$$\left(-\frac{q^2}{h^3} + \frac{\Lambda^{2p} h^{2p+1}}{(p+2)^2} + \frac{\delta}{\beta} \right) \partial_x h - \delta \frac{2p\Lambda^{2p-1} h^{2p+2}}{(2p+3)(p+2)^2} \partial_x^2 h = \frac{1}{\beta} \left(\lambda - \frac{1}{h} \left(\frac{q}{h \left(C + \frac{h}{p+2} \right)} \right)^{\frac{1}{p}} \right)$$

with Λ defined by (49).

We introduce the space slow variable $\xi = \eta x$, with η small (strictly positive). This classical assumption of slow space variations makes negligible the higher-order terms $\partial_{xx}^2 h$ and $(\partial_x h)^2$. Then the previous equation writes:

$$\boxed{\left(1 - \frac{\beta}{\delta} \left(\frac{q^2}{h^3} - \frac{\lambda^{2p} h^{2p+1}}{(p+2)^2} \right) \right) \partial_x h = \frac{1}{\delta} \left(\lambda - \frac{1}{h} \left(\frac{q}{h \left(C + \frac{h}{p+2} \right)} \right)^{\frac{1}{p}} \right) + O\left(\eta^2\right)} \tag{68}$$

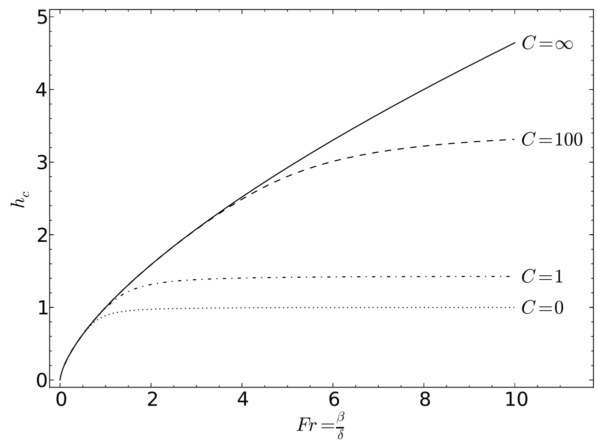


Figure 5: Critical-depth line h_c in function of the Froude number Fr for different values of the slip coefficient C defined by (70). The curves (value $h_c = 1$) separates sub-critical and supercritical flows.

In the hydraulic context, it is so-called the *gradually varied flow equation* (if neglecting the second order terms in η). The left-hand side and the right-hand side of equation (68) define two heights. The normal-depth line $h_N$is the solution of the vanishing right-hand side. It is the uniform solution of the steady-state equation (60).
The critical-depth line h_c is the solution of the vanishing left-hand side. Since $\frac{\beta}{\delta} = Fr^2$, the left-hand side can also be written in the form $\left(1 - F_h^2\right) \partial_x h$ where F_h is a dimensionless number similar to Fr but which depends on h.
Let us set the constant flow rate $q = 1$. The relation (28) allows to simplify the expression of the "hydraulic" Froude number F_h as follows:

$$F_h^2 = \frac{Fr^2}{h^3}\left(1 - \frac{h^{2(p+2)}}{\left(C\left(p+2\right)+1\right)^2}\right) \tag{69}$$

The critical-depth line h_c defines a critical hydraulic Froude number such that:

$$F_h\left(h_c\right) = 1 \tag{70}$$

For large Froude numbers, we have $h_c^{p+2} \approx 1 + C\left(p+2\right)$, thus the critical-depth line h_c only depends on the slip coefficient. At low Froude numbers or for large slip coefficient values the critical height value $h_c \approx Fr^{\frac{2}{3}}$.

By analogy with a hydraulic description, if $F_h < 1$ (i.e. $h > h_c$) the flow is called sub-critical; if $F_h > 1$ (i.e. $h < h_c$) the flow is called supercritical.

The value of the critical height h_c is plotted on figure 5 for several values of the slip coefficient. The value $h_c = 1$ separates sub-critical and supercritical flows.

In summary, we have derived a two-equation model (shallow-water type model), see (67), for multi-regime flows but not fully consistent with the free-surface Navier-Stokes equations since the basal shear stress has been derived at order 0 only (see discussion above). Nevertheless, the latter allows to derive the corresponding gradually varied flow equation (68), hence define hydraulic-like Froude number and corresponding sub-critical, supercritical flows defined in function of (α, β, δ) and C. These new expressions and the gradually varied flow model will be illustrated numerically in Section 12.

10. Consistent multi-regime two-equation models

In this section, we derive two-equation models, consistent with the free-surface Navier-Stokes equations and valid for all viscous regimes. Recall that two-equation models presents few advantages compared to

one-equation models (see the previous discussion).
To obtain a first version of a consistent two-equation model, the higher-order terms ($h.o.t.$) in the R.H.S. of Eqn (67) are expanded; it is the basal friction higher-order terms. The resulting model, which is in a conservative form, is consistent and globally 0th order but it is linearly ill-posed (i.e. the model linearized in the vicinity of the constant uniform solution, flat free surface is ill-posed). Next, the second equation (the momentum equation) is written in a quasi-linear form. Keeping the classical structure of Saint-Venant equations (for the common terms), a new family of multi-regime model is obtained, see (79)(80). They can be interpreted as three-equation models. This last structure lead to a linearly well-posed model (in addition to be consistent with the free-surface Navier-Stokes equations and valid for all viscous regimes).
The present study does not detail the mathematical analysis of these formally derived models, nor the numerical analysis. It will be done in a forthcoming study.
Nevertheless, it seems that in the case of Regime C (and in Regimes A and B in the newtonian case), it should not have any particular difficulty to write the numerical analysis, compared to the standard Newtonian shallow-water equations. On the contrary, in the general case (power-law, regimes A or B), the extensive analysis is not straightforward; it would deserve to be addressed into more details.

The outline of the present section is as follows. First, the general structure of two-equation models is presented (both in the Prandtl coordinate system and in the mean-slope coordinate system). Next, the expressions of each term at the order required are derived; it leads to the unified two-equation model (74)(78). This model is consistent with the primitive free-surface Navier-Stokes flow equations but a standard linear analysis (in the vicinity of the constant uniform solution, flat free surface) demonstrates that it is linearly ill-posed. Next, the model is re-formulated as a three-equation like model in a quasi-linear form, see (79)-(80); it is based on the natural new variable Λ introduced previously. This last model is shown to be linearly well-posed.

10.1. Integral form (in Prandtl's coordinate system)

In this section, we consider the Prandtl coordinate system, see AppendixA. The average momentum equation writes:

$$\begin{aligned}
&\partial_t q + \partial_x \Big(\int_{-h}^{0} \hat{u}^2 \mathrm{d}\hat{z}\Big) + \frac{\delta}{\beta} \int_{-h}^{0} \partial_x (\hat{\pi} - \alpha \hat{\sigma}) \mathrm{d}\hat{z} \\
&+ \frac{\delta}{\beta} (\hat{\pi}_{|\hat{z}=-h} - \alpha \hat{\sigma}_{xx|\hat{z}=-h}) \partial_x H - a \hat{u}_{|\hat{z}=0} = \frac{1}{\beta} (\lambda h - \hat{\sigma}_{xz|\hat{z}=-h})
\end{aligned} \tag{71}$$

Let us specify the order required for each term in order to derive a consistent order 0 two-equation model.

- Case of Regime C. The three parameters $((\beta, \alpha, \delta)$ are supposed to be small, furthermore let us them supposed to be of the same order of magnitude. Then, we need to derive the inertial term, the viscous term and the pressure term at order 0; while the shear-stress term at bottom in the R.H.S. must be derived at order 1. The basic equation writes:

$$\begin{aligned}
&\partial_t q + \partial_x \Big(\int_{-h}^{0} (\hat{u}^{(0)})^2 \mathrm{d}\hat{z}\Big) + \frac{\delta}{\beta} \int_{-h}^{0} \partial_x \hat{\pi}^{(0)} \mathrm{d}\hat{z} \\
&+ \frac{\delta}{\beta} \hat{\pi}^{(0)}_{|\hat{z}=-h} \partial_x H - a \hat{u}^{(0)}_{|\hat{z}=0} = \frac{1}{\beta} (\lambda h - \hat{\sigma}^{(1)}_{xz|\hat{z}=-h}) + \mathcal{O}\big(\tfrac{(\alpha+\beta+\delta)^2}{\beta}\big)
\end{aligned} \tag{72}$$

  Hence, in order to obtain the two-equation model (order 0) for Regime C, it remains to calculate the basal shear stress $\hat{\sigma}^{(1)}_{xz|\hat{z}=-h}$ order 1. It is done in next subsection.

- Case of Regime A ($\delta = \mathcal{O}(1)$). We need to derive the inertial and viscous terms at order 0, the shear-stress term at bottom in the R.H.S. at order 1, and the pressure term at order 1. The basic equation writes:

$$\partial_t q + \partial_x(\int_{-h}^{0} (\hat{u}^{(0)})^2 \mathrm{d}\hat{z}) + \frac{\delta}{\beta}\int_{-h}^{0} \partial_x \hat{\pi}^{(1)} \mathrm{d}\hat{z}$$

$$+\frac{\delta}{\beta}\hat{\pi}^{(1)}_{|\hat{z}=-h}\partial_x H - a\hat{u}^{(0)}_{|\hat{z}=0} = \frac{1}{\beta}(\lambda h - \hat{\sigma}^{(1)}_{xz|\hat{z}=-h}) + \mathcal{O}(\tfrac{(\alpha+\beta)^2+\varepsilon^2}{\beta})$$

- Case of Regime B ($\alpha = \mathcal{O}(1)$). It is the same as Regime C: we need to derive the inertial term the viscous term and the pressure term at order 0 and the shear-stress term at bottom in the R.H.S. at order 1. The basic equation is (72).

10.2. The advective, pressure and viscous terms (in Prandtl's coordinate system)

The expressions of the advective, pressure and viscous terms are the same as those presented in the previous section. Concerning the advective term, since we seek to preserve the relation:

$$\int_{-h}^{0} (\hat{u})^2 \mathrm{d}\hat{z} \approx \int_{-h}^{0} (\bar{u})^2 \mathrm{d}\hat{z} = h\bar{u}^2 = \frac{q^2}{h}$$

We write it as follows:

$$\int_{-h}^{0} (\hat{u}^{(0)})^2 \mathrm{d}\hat{z} = \frac{q^2}{h} + \frac{\Lambda^{2p} h^{2p+3}}{(p+2)^2(2p+3)}$$

In the Prandtl's coordinate system, the other terms write as follows:

$$\frac{\delta}{\beta}(\int_{-h}^{0} \partial_x \hat{\pi}^{(1)} \mathrm{d}\hat{z} + \hat{\pi}^{(1)}_{|\hat{z}=-h}\partial_x H) = h\partial_x H + \mathcal{O}(\frac{\varepsilon^2}{\beta})$$

10.3. A first form of consistent two-equation model

10.3.1. The model structure

Let us recall that the two-equations model in integral form is (64). Then to obtain a consistent unified two-equation model, it remains to derive an expression of the friction term :

$$\mathrm{T}_{\text{fric}} = \lambda h - \sigma^{(1)}_{xz|b}$$

at order 1 for each regime (unified expression). This is the crucial term.

In Equation (71), the friction term T_{fric} is dominating either if β is very small or if the flow is uniform. Recall that in the uniform case, it is natural to seek to recover the reference flow at equilibrium i.e. $q^{(0)}$ (see the discussion led in the introduction of the previous section).

Let us derive the basal shear rate $\sigma_{xz|b}$ at order 1. We seek to keep the same structure as its order 0 expression, see (66). Hence, we seek an expression of $\sigma_{xz|b}$ at order 1 as follows:

$$\sigma^{(1)}_{xz|b} = \mathrm{sgn}(\Lambda)\left(\frac{|q|}{h(C+\frac{h}{p+2})}\right)^{\frac{1}{p}} + \beta\tau^{(\beta)} \tag{73}$$

where $\tau^{(\beta)}$ is the corrective term to be determined.

If going back to the mean-slope coordinate system (and considering the mass balance term a), the two-equation model writes:

$$\boxed{\begin{cases} \partial_t h + \partial_x q &= a \\ \partial_t q + \partial_x \left(\dfrac{q^2}{h} + P(h,\Lambda)\right) &= \dfrac{1}{\beta}\left(\Lambda h - \mathrm{sgn}(\Lambda)\left(\dfrac{|q|}{h(C+\frac{h}{p+2})}\right)^{\frac{1}{p}}\right) - \tau^{(\beta)} \\ & \quad + a\Lambda|\Lambda|^{p-1}h^p\left(C + \dfrac{h}{p+1}\right) \end{cases}} \tag{74}$$

where Λ is defined by (49), $\tau^{(\beta)}$ is the basal shear rate corrective term determined below, and $P(h,\Lambda)$ denotes the correction of the advective term defined by:

$$P(h,\Lambda) = \frac{\Lambda^{2p}h^{2p+3}}{(p+2)^2(2p+3)} \tag{75}$$

.

10.3.2. The basal friction corrective term $\tau^{(\beta)}$

Using the definition of ϕ in(51), we define $q^{(1,\beta)}$ as follows:

$$q^{(1,\beta)} = \frac{1}{\beta}\left(q^{(1)} - q^{(0)}\right)$$

Then we have:

$$\left(\frac{|q|}{h(C+\frac{h}{p+2})}\right)^{\frac{1}{p}} = \left(\frac{|q^{(0)}|}{h(C+\frac{h}{p+2})}\right)^{\frac{1}{p}}\left(1 + \beta\frac{1}{p}\frac{q^{(1,\beta)}}{q^{(0)}}\right) + \mathcal{O}((\sum_{\phi\in\Phi}\phi)^2 + \varepsilon^2)$$

Combining with (73), we obtain:

$$\sigma^{(1)}_{xz|b} = \sigma^{(0)}_{xz|b}\left(1 + \beta\frac{1}{p}\frac{q^{(1,\beta)}}{q^{(0)}}\right) \tag{76}$$

with $\sigma^{(0)}_{xz|b}$ defined by (66). Therefore, the corrective term $\tau^{(\beta)}$, order 1 both in small parameters and in the geometrical ratio, writes:

$$\tau^{(\beta)} = \frac{1}{p}\sigma^{(0)}_{xz|b}\frac{q^{(1,\beta)}}{q^{(0)}} \tag{77}$$

Next, we develop the terms $q^{(*)}$ (see AppendixB for their expressions), and the following expression of the basal friction corrective term follows:

$$\tau^{(\beta)} = B_1(h,\Lambda)\partial_x h + B_2(h,\Lambda)\partial_x\Lambda + B_3(h,\Lambda)\partial_t\Lambda + B_4(h,\Lambda)\partial_x b + B_5(h,\Lambda)a \tag{78}$$

where the coefficients B_k are given in AppendixC.

Let us recall that:

$$\partial_x\Lambda = -\delta\partial^2_{xx}H \text{ and } \partial_t\Lambda = -\delta\partial^2_{xx}q$$

Since $\partial_x\Lambda = O(\delta) = \partial_t\Lambda$, in the case of Regime B and C (δ is small), the terms in B_2 and B_3 are negligible.
The topography variation $\partial_x b$ appears a-priori in all terms $B_\square$ since the definition of Λ.

The resulting two-equation model (74)(78) is consistent with the primitive free-surface Navier-Stokes flow equations, it is valid for any viscous regime (regimes A, B and C), for any mean-slope value and with

local varying topography (if respecting the few limitations mentioned previously in particular in terms of compatibility between λ and the slip coefficient C).
Nevertheless, a standard linear analysis (in the vicinity of the constant uniform solution, flat free surface) demonstrates that this system (74)(78) is linearly ill-posed. The proof which is based on the classical technics, see e.g. [5], is not detailed here. To circumvent this crucial problem, in next section we derive a new form of the momentum equation which is shown to be linearly well-posed.

10.4. An unified model, consistent and linearly well-posed

First, it is noticed that the new variable Λ defined by (49), is the natural variable to derive all unified expressions and models (i.e. the expressions and equations valid for all the regimes considered). Second, in view to analyze the linear stability of the model, the momentum equation is written in a quasi-linear form.

The momentum equation of the two-equation model (74) can be written as follows:

$$\begin{aligned} &\partial_t q + \partial_x \left(\frac{q^2}{h} + P(h,\Lambda) \right) + B_1(h,\Lambda)\partial_x h + B_2(h,\Lambda)\partial_x \Lambda \\ &= \frac{1}{\beta} \left(\Lambda h - \operatorname{sgn}(\Lambda) \left(\frac{|q|}{h(C + \frac{h}{p+2})} \right)^{\frac{1}{p}} \right) - B_3(h,\Lambda)\partial_t \Lambda - B_4(h,\Lambda)\partial_x b \end{aligned}$$

with the coefficients B_k defined in AppendixC. (For a sake of clarity, the mass balance term a has been skipped).

10.4.1. Generalized quasi-linear form of the model

The last expression above suggests to consider the shallow-water system in the following three-equation quasi-linear form:

$$\boxed{\left\{ \begin{aligned} &\partial_t h + \partial_x q = 0 \\ &\partial_t q + \mathcal{A}(h,q,\Lambda)\partial_x h + \mathcal{B}(h,q,\Lambda)\partial_x q + \mathcal{C}(h,q,\Lambda)\partial_x \Lambda = \frac{G_0}{\beta} \left(\Lambda h - \operatorname{sgn}(\Lambda) \left(\frac{|q|}{h(C + \frac{h}{p+2})} \right)^{\frac{1}{p}} \right) \\ &\quad + \mathcal{D}(h,q,\Lambda)\partial^2_{xx} q + \mathcal{E}(h,q,\Lambda)\partial_x b \end{aligned} \right.} \tag{79}$$

with:

$$\boxed{\partial_t \Lambda = \delta \partial^2_{xx} q} \tag{80}$$

and the extra initial condition: $\Lambda(0)$ given. G_0 is an arbitrary constant to be determined (it is discussed later).

Keeping the classical structure of Saint-Venant equations, we set:

$$\left\{ \begin{array}{lcl} \mathcal{A}(h,q,\Lambda) & = & -\left(\frac{q}{h}\right)^2 + c^2 \\ \mathcal{B}(h,q,\Lambda) & = & 2\frac{q}{h} \end{array} \right. \tag{81}$$

with c the wave velocity.
Let us point out that in the classical case (Newtonian case, Regime C), it has been proved in [42] that this choice of $\mathcal{A}$ and $\mathcal{B}$ in the Saint-Venant model formulation leads to solutions invariant by translation (Galilean invariance). In the present extended case, the wave velocity c depends a-priori on the two variables (h, Λ).

Straightforward calculations demonstrate that (74)(78) is formally equivalent (at order 1 in β) to (79)(80) if setting:

$$\left\{\begin{array}{rcl} \mathcal{C}(h,q,\Lambda) & = & \left(C(p-1)+\frac{p}{p+2}h\right)\left(C+\frac{1}{p+2}h\right)p\Lambda^{2p-1}h^{2p+1} \\ \mathcal{D}(h,q,\Lambda) & = & \left((C+\frac{1}{p+2}h)\delta - G_0\frac{(p+2)(2p+3)C^2+2(2p+3)hC+2h^2}{(2p+3)(h+(p+2)C)}\right)p\Lambda^{p-1}h^{p+1} \\ \mathcal{E}(h,q,\Lambda) & = & G_0\frac{(p+2)}{p((p+2)C+h)}\frac{1}{h^p\Lambda^{p-1}}A_4(h,\Lambda)\partial_x b \end{array}\right. \tag{82}$$

with $A_4(h,\Lambda)$ defined in AppendixB,

$$c^2(h,\Lambda) = \left(p^2C^2 + 2\frac{p(p+1)hC}{(p+2)} + \frac{h^2(p+1)^2}{(2+p)^2} - \frac{(h+pC)((2p^2+7p+6)C^2+2(2p+3)hC+2h^2)}{(2p+3)(h+(p+2)C)}pG_0\right)h^{2p}\Lambda^{2p} \tag{83}$$

and: $G_0 = 1$.

From now, we consider the model defined by (79)-(83), with the constant G_0 to be determined.

10.4.2. Well-posedness conditions

In this section, a classical linear analysis, see e.g. [5], of this new quasi-linear system (79)-(83) is led; we refer to [42] for a similar analysis but in the mono-regime C case. The system is classically linearized in the vicinity of the constant uniform solution, flat free surface. A Fourier analysis is the basic mathematical tool to analyze the well-posedness of the resulting system. In the standard newtonian case, this linearized system would lead to the Orr-Sommerfeld equation, see e.g. [5, 6].
The two properties sought are the following:
i) the first order part of the system (the LHS of the momentum equation) remains hyperbolic;
ii) the second order term remains dissipative.

These two properties are satisfied if the following two are satisfied (see (79) and (82)):

$$c^2(h,\Lambda) > 0 \text{ and } \mathcal{D}(h,\Lambda) \geq 0 \tag{84}$$

It can be shown that under these two conditions (84), the linearized system is stable, including for high frequency perturbations. Straightforward calculations demonstrate that (84) is equivalent to the following conditions on the constant G_0:

$$\left\{\begin{array}{rcl} G_0 & < & \dfrac{\left(p^2C^2+2h\frac{p(p+1)}{2+p}C+h^2\frac{(p+1)^2}{(2+p)^2}\right)(2p+3)(h+(p+2)C)}{p(h+pC)((2p^2+7p+6)C^2+2(2p+3)hC+2h^2)} \\ G_0 & \leq & \dfrac{(2p+3)(h+(p+2)C)\left(C+h\frac{1}{p+2}\right)}{(p+2)(2p+3)C^2+2(2p+3)hC+2h^2)}\delta \end{array}\right. \tag{85}$$

Remark 26. In the case δ small (regimes B and C), the quasi-linear form of the model can be formulated without the viscous term $\mathcal{D}(h,q,\Lambda)$, then the second inequality in (84) does not have to be satisfied.

Remark 27. In the no-slip case ($C=0$), the two conditions (85) simplify as follows:

$$\left\{\begin{array}{rcl} G_0 & < & \dfrac{(2p+3)(p+1)^2}{2p(p+2)^2} \\ G_0 & \leq & \dfrac{(2p+3)}{2(p+2)}\delta \end{array}\right. \tag{86}$$

Hence in the case $C = 0$ and $G_0 = 1$, (86) is satisfied if and only if the power-law exponent p satisfies: $p < \sqrt{3}$. This result demonstrates that the two-equation model (74)(78) with $G_0 = 1$, is linearly well-posed if and only $p < \sqrt{3}$, which includes the Newtonian case ($p = 1$) but not the classical higher-values of p (e.g. $p \geq 2$ or $p = 3$, Glen's law used for ice). On the contrary, given p, if G_0 satisfies (86) then the resulting model is linearly well-posed in a general case in terms of the power-law exponent p.

In summary, under these two conditions (85), the unified model (79)-(83) is a-priori a good model since linearly well-posed; also it is consistent with the primitive free-surface Navier-Stokes flow equations and valid for any viscous regime (regimes A, B and C) (and if respecting the few limitations mentioned previously in particular in terms of compatibility between λ and the slip coefficient C).

The mathematical and numerical analyses of the present new models would deserve to be detailed; it will be done in a forthcoming manuscript.

In next section, we show that the unified model (79)(80) contains existing models from the literature if considering the same particular cases.

11. Comparison to the literature

In this section, we show that the one-equation models at order 0 (58) and at order 1 (61), the two-equation model (74) contain all terms of existing models from the literature. To do so, we have to consider potentially a vanishing mean slope ($\lambda = 0$) (depending on the references), a flat bottom ($b = 0$ hence $H = h$), the same regime (Regime C) at same order (order 1), and the same basal condition (generally no-slip except in one case an asymptotically vanishing friction).

11.1. Thin-film flows literature - Power-law case

If considering the present unified one-equation model order 1 (61) in the particular case: Regime C, a non-vanishing mean slope but a flat bottom ($b = 0$ hence $H = h$), adherence at bottom ($C = 0$), and $a = 0$, then we recover the equation stated in [18] (see also [19]), apart from the surface tension term in κ since it is not considered in the present study.

Under these assumptions, the unified one-equation model (61) simplifies greatly and writes:

$$\partial_t h + \partial_x[\frac{(\lambda h)^p}{p+2} h^2 (1 - p\frac{\delta}{\lambda}\partial_x h) + \beta \frac{2p^2}{(p+2)(2p+3)} \lambda^{(3p-1)} h^{3(p+1)} \partial_x h] = 0$$

It is the same equation as the one-equation model stated in [18] (see also [19]) since the latter writes:

$$\begin{aligned} &\partial_t h + \partial_x \left(\frac{n}{2n+1} h^2 \left(\tilde{\lambda} h\right)^{1/n} \right) = \\ &\partial_x[\left(\frac{\tilde{\delta} c}{n \tilde{\lambda}} h^2 \left(\tilde{\lambda} h\right)^{1/n} \frac{n}{2n+1} - \beta \frac{2}{(2n+1)(3n+2)} \tilde{\lambda}^{3/n - 1} h^{3(1/n - 1)} \right) \partial_x h] - \partial_x (\kappa \frac{\tilde{\delta}}{n\tilde{\lambda}} \partial_x^3 h) \end{aligned} \tag{87}$$

with $n = \frac{1}{p}$ and the dimensionless numbers $\tilde{\lambda}$, $\tilde{\delta}$ defined slightly differently.

Next, let us write the second equation of the unified model (74) (the momentum equation) under the same assumptions. Under these assumptions, the corrective friction term defined in (78) writes:

$$\tau^{(\beta)} = -\frac{p}{(p+2)(2p+3)} (\lambda H)^{2p} H^2 \partial_x H$$

Then, the present unified model simplifies as follows:

$$\partial_t q + \partial_x \left(\frac{q^2}{H} + \frac{1}{2}\frac{\delta}{\beta} H^2 - \frac{p}{(p+2)(2p+3)^2} (\lambda H)^{2p} H^3 \right) \quad = \quad \frac{1}{\beta} \left(\lambda H - \left((p+2) \frac{q}{H^2} \right)^{\frac{1}{p}} \right) \tag{88}$$

The model derived in ([19], Eqn (47)) is recovered. Furthermore, if defining differently the term $\mathcal{A}$ in (81) and with the corresponding constant G_0, then the model derived in ([42], Eqn (3.46)) is recovered too.

11.2. Thin-film flows literature - Newtonian case

The classical Benney equation [4] is a first order one-equation model, taking into account surface tension. This equation writes as follows, see e.g. [5]:

$$\partial_t h + \partial_x \left(h^3 + \left(\frac{6}{5} Re h^6 - \cot\theta h^3 \right) \partial_x h + W h^3 \partial_x^3 h \right) = 0 \tag{89}$$

with W the Weber number.

Using the average velocity as the characteristic velocity and by setting $p = 1$ (Newtonian fluid) we obtain: $\lambda = 3$, see (28). Next, under the right assumptions, the flow rate writes:

$$q^{(1)}_{(p=1,C=0,\lambda=3,\delta=\varepsilon\lambda\cot\theta)} = h^3 + \varepsilon \left(\left(\frac{6}{5} R_e h^3 - \cot\theta \right) h^3 \partial_x h - \frac{5}{8} R_e h^4 a \right)$$

Therefore, under the right assumptions, the present unified one-equation model degenerates to the Benney equation (excepted the surface tension term since it is not considered here).

In the case of no-slip and a newtonian fluid, than the corrective friction term defined in (78) writes:

$$\tau^{(\beta)}_{(C=a=0,p=1)} = \frac{1}{15} \Lambda^2 h^4 \partial_x h + \frac{1}{105} |\Lambda| h^5 \partial_x \Lambda - \frac{1}{60} h^4 \partial_t \Lambda$$

Hence, in the case of Regime C, $\tau^{(\beta)}$ simplifies as follows:

$$\tau^{(\beta)} = \frac{1}{15} \lambda^2 h^4 \partial_x h$$

And the unified model (74) simplifies as follows:

$$\begin{cases} \partial_t h + \partial_x q = 0 \\ \partial_t q + \partial_x \left(\frac{q^2}{H} + \frac{1}{2}\frac{\delta}{\beta} H^2 + \frac{2}{225} \lambda^2 H^5 \right) = \frac{1}{\beta} \left(\lambda H - 3 \frac{q}{H^2} \right) \end{cases}$$

While the momentum equation stated in ([43], p34) writes:

$$\begin{cases} \partial_t h + \partial_x q = 0 \\ \partial_t q + \partial_x \left(C_1 \frac{q^2}{H} + \frac{1}{2}\frac{\delta}{\beta} H^2 + (\frac{1}{5} - \frac{C_1}{9} - \frac{6}{75}) \lambda^2 H^5 \right) = \frac{1}{\beta} \left(\lambda H - 3 \frac{q}{H^2} \right) \end{cases} \tag{90}$$

with the constant C_1 almost arbitrary, see [43].

Hence the equation stated in [43] is retrieved (excepted the surface tension term which is not considered here). Furthermore, it has been proved in [43] (using some algebra calculations) that the two-equation model presented in [43] is identical to those derived in [10] if using the order 0 flow rate - depth relation : $q^{(0)} = \frac{1}{3}\lambda h^3$.

Remark 28. The result shown in [33] is the following: any regular Navier-Stokes solution is solution of the present two-equation model (in the case of no-slip at bottom, Regime C with $Re \sim Fr \sim O(1)$) modulo a remainder in $o(\varepsilon)$ (i.e. the RHS in the form $\frac{1}{\beta}(...) + o(\varepsilon)$). The result has been obtained in the particular case $C_1 = \frac{6}{5}$ but the proof remain valid for any model in the parameter form (90). .

Asymptotically vanishing friction case. The present unified two-equation model includes the order one terms of the model derived in [7]. The simplifications considered in [7] are : a vanishing mean slope and flat bottom ($\lambda = 0$ and $b = 0$), Regime C ($\alpha = \beta = \delta = \varepsilon$), and an asymptotically vanishing friction at bottom i.e. we have: $C = \dfrac{1}{\varepsilon\kappa}$, ε the geometrical ratio, and $\varepsilon\kappa$ the friction coefficient. In this specific case, the corrective friction term defined in (78) becomes negligible. Then, the momentum equation of the unified model (74) writes:

$$\partial_t q + \partial_x \left(\frac{q^2}{H} + \frac{\delta}{\beta} \frac{1}{2} H^2 \right) = - \frac{\kappa}{\left(1 + \varepsilon \frac{\kappa H}{3}\right)} \frac{q}{H}$$

Therefore the present unified model contains all the first order terms derived in [7]. The goal of [7] is to derive a higher-order viscous model (in the particular regime flow detailed above), hence the model derived is order 1 in ε (i.e. in $\mathcal{O}\left(\varepsilon^2\right)$). The extra viscous term order 1 is not included into the present unified model since we did not derive the model up to order two (using the present terminology). Let us point out that for the more general configurations considered in the present article (all range of basal friction, multi-regimes, mean slope not vanishing, non flat bottom), the fully consistent second order model would be extremely complex. Nevertheless, under the assumptions made in [7], the extra term order 1 can be obtained quite easily using the method presented in Section 4.3 too.

11.3. Glacier flow models

In the glaciology literature, the classical Shallow Ice Approximation (SIA) is derived with no-slip boundary condition at bottom, in the horizontal-vertical coordinate system ($\lambda = 0$), and with the rheology exponent $p = 3$ (Glen's law). As mentioned previously, if modeling ice flows the inertial term can be neglected, hence we set : $\beta = 0$. The SIA equation is known to be valid for high shear rate flow, corresponding to the "ice-sheet" case i.e. Regime A, see Table 1. Classically, it is derived by assuming *a-priori* scalings on the normal stresses ($\sigma_{xx} = \sigma_{zz} = -\pi$) and under the assumption of negligible topography variations, see e.g. [2]. The SIA equation writes, see e.g. [2, 22, 24] :

$$\partial_t h - \partial_x (\Gamma \frac{h^{p+2}}{p+2} \partial_x H |\partial_x H|^{p-1}) = a$$

with Γ a constant depending on μ_0 and ρg. Therefore, Equation (58) includes the SIA model since for $\lambda = 0$ (vanishing mean-slope), Regime A, no-slip at bottom, it writes:

$$\partial_t h - \delta^p \partial_x (\frac{h^{p+2}}{p+2} \partial_x H |\partial_x H|^{p-1}) = a$$

with $\delta = \frac{\rho g H^{n+2}}{\mu_0 U^n L}$.

In the case of a non-vanishing velocity at bottom imposed (Dirichlet's condition $u = u_b$ at bottom), and under the assumption of non "abrupt" ice surface variations, the SIA model has been extended in [27] (Chapter 10 p 633) as follows:

$$\partial_t h - \partial_x (\frac{h^{p+2}}{p+2} \partial_x H |\partial_x H|^{p-1} + u_b h) = a \tag{91}$$

Equation (58), for $\lambda = 0$, Regime A, friction at bottom, writes:

$$\partial_t h - \delta^p \partial_x (\frac{h^{p+2}}{p+2} \partial_x H |\partial_x H|^{p-1}) - C\delta^p \partial_x H |\partial_x H|^{p-1} h^{p+1} = a \tag{92}$$

Using (45), the expression of u_b at order 0 writes:

$$u_b^{(0)} = -C\delta^p h^p \partial_x H |\partial_x H|^{p-1} \tag{93}$$

If replacing in Equation (92), we obtain:

$$\partial_t h - \delta^p \partial_x \left(\frac{h^{p+2}}{p+2} \partial_x H |\partial_x H|^{p-1} + h u_b^{(0)} \right) = a$$

Thus, Equation (58) includes the SIA model derived in [27], in the case velocity at bottom u_b imposed (if setting $\delta = 1$, hence a particular Regime A).

Note that a similar equation to (91) is considered in [30], and it is shown formally that it is valid for all friction - slip regimes. To do so, the authors introduce a slip ratio R (R tends to 0 if pure slip at bottom (pseudo-plug flow) and R tends to 1 if no-slip at bottom).

Observe that in the latter references, the shallow models are derived based on a-priori scaling and with a-priori weak variations of topography and/or free-surface. Also, the asymptotic are not done fully in the non-linear terms (eg the power-law term depth-integrated).

In [28], the authors introduce classically an *a-priori* double scaling on the velocity and the stress components, leading to one particular Regime A (case $\lambda = 0$ and $\delta = 1$). Next, the authors obtain the equations (43)-(44) for this particular case, and the exact solutions order 0. The sketch of the derivation of the higher-order terms (hence higher order shallow ice models) up to order 2 in the geometrical ratio ε is presented (still in the same case). Almost all fields are calculated at order 2 in ε excepted those singular (i.e. the normal stress components and the pressure), see [28] section "second order stress formulas".

In summary, both the unified one-equation model (61) and the unified two-equation model (74) includes all the classical equations mentioned above, if considering a vanishing mean slope ($\lambda = 0$), a flat bottom ($b = 0$ hence $H = h$), the same regime (Regime C) at same order (order 1), and the same basal condition (generally no-slip except in one case an asymptotically vanishing friction). Roughly, compare to the classical shallow models from the literature, the present unified models contain extra terms in $\delta, \beta\delta$ and in C (the slip coefficient), and remain valid for non-vanishing mean slopes, non-flat topographies and multi-regimes of course.

12. Numerical examples: two-regime flows with sharp transition

In this section we illustrate numerically the capabilities of the unified model (58) by performing the multi-regime numerical solution in two cases; both presenting sharp but continuous changes of regime.

In Case 1 the change of regime is due to the change of the bottom boundary condition.

In Case 2 the change of regime is due to a change of the topography (presented here in term of mean slope).

In both cases the change of the boundary condition (resp. the mean slope) are sharp but continuous. The goal is to capture accurately this multi-regime flow (with a sharp change) using the single one-equation model (58).

12.1. Solution computed

We consider the steady state solution; it satisfies (60). Assuming the fluid flows from left to right (i.e. $q_0 > 0$), the steady solution writes:

$$\partial_x H = \partial_x h + \partial_x b = \frac{1}{\delta} \left(\lambda - \frac{1}{h} \left(\frac{1}{h\left(C + \frac{h}{p+2}\right)} \right)^{1/p} \right) \tag{94}$$

At the left part of the domain, let us assume the equilibrium is reached, that is to say a steady state with a constant height. We use this equilibrium to define the reference flow: $q_0 = 1$, $h_0 = 1$, $C = C_0$. λ is defined by the relation (28) that means $\lambda = \left(C_0 + \frac{1}{p+2}\right)^{-1/p}$.

We want to consider the transition between two regimes occurring when either the boundary condition changes (from adherence to slip) or the mean slope changes.

Let us point out we have fixed the reference flow. Thus we obtain a set of dimensionless parameters. If we choose another reference height, we obtain a different set of dimensionless parameters. This "change of reference basis" can be expressed in terms of the previous dimensionless parameters through the following relations:

$$\alpha^{new}(h) = \frac{\alpha}{h^{1+\frac{2}{p}}} \qquad \delta^{new}(h) = h^{1+\frac{2}{p}}\delta \tag{95}$$

If $h = 1$ then $\delta^{new} = \delta$ and $\alpha^{new} = \alpha$. If we have $h \neq 1$, then it is a new regime.

Any equilibrium solution satisfies (94). We expect to reach an equilibrium at infinity: $h = h_\infty$, $\partial_x b = \tan\theta_\infty$, $C = C_\infty$; therefore equation (94) writes:

$$h_\infty^{p+1}\left(C_\infty + \frac{h_\infty}{p+2}\right) = \frac{1}{(\lambda - \delta\tan\theta_\infty)^p} \tag{96}$$

As mentioned before we consider two cases:

- Case 1: the mean slope is constant, hence $\tan\theta_\infty = 0$ and the change of regime is due to a change of the bottom boundary condition ($C_0 \neq C_\infty$). The equilibrium at infinity is given by:

$$h_\infty^{p+1}\left(C_\infty + \frac{h_\infty}{p+2}\right) = C_0 + \frac{1}{p+2} \tag{97}$$

  Case 1 is separated in two sub-cases:

  - Case 1a: constant mean slope unperturbed $b(x) = 0$.
  - Case 1b: constant mean slope with arbitrary varying topography $b(x) \neq 0$.

- Case 2: the bottom boundary condition is constant ($C_0 = C_\infty$). The change of regime is due to a change of mean slope $\tan\theta_\infty \neq 0$. The equilibrium at infinity is given by:

$$h_\infty^{p+1}\left(C_0 + \frac{h_\infty}{p+2}\right) = \frac{1}{\left(\left(C_0 + \frac{1}{p+2}\right)^{-1/p} - \delta\tan\theta_\infty\right)^p} \tag{98}$$

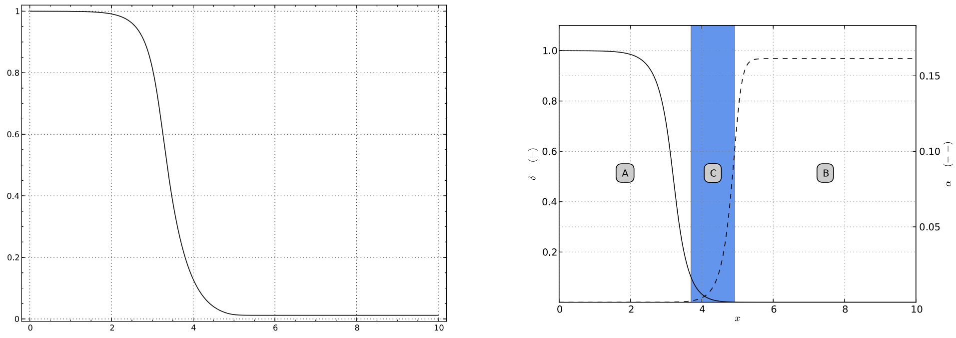
Then we compute the corresponding height solution of the unified model (94).

Also, we show that the steady-state multi-regime solution (solution of (94)) corresponds to a change of regime in the sense sub-critical to supercritical flow (see calculation presented in Section 9).

Below, the steady unified model (94) is solved numerically using a classic RK4 method.

12.2. Case 1: Multi-regime transition due to a change of the bottom boundary condition

Let us consider a domain in two parts with a sharp but continuous transition between them. We set the geometric scaling parameter ε=10^{-2}, the power-law exponent $p = 3$. At the left half of the domain ($x < 4$), we set $\gamma = \varepsilon$ and $Fr^2 = \varepsilon$ in order to obtain a Regime A ($\alpha = \varepsilon^2$ and $\delta = 1$) with adherence at bottom ($C = 0$). At the right half of the domain part ($6 < x$), we consider a slip boundary condition with $C = 10^7$ with a smooth transition ($4 \leq x \leq 6$) (see figure 6a).



(a) Flow height $h(x)$ (b) $\alpha(h)$ and $\delta(h)$

Figure 6: Case 1-a: Multi-regime transition due to a change of the bottom boundary condition

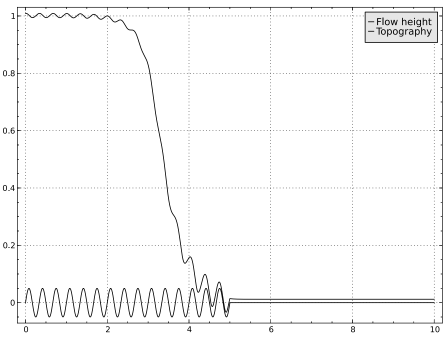


Figure 7: Case 1-b: Flow height and topography of the multi-regime transition over an arbitrary topography

Case 1a.

The local topography variations $b(x)$ are null (non-horizontal flat bottom).

We solve numerically the steady unified model (94) using a classic RK4 method. The resulting flow height is plotted on figure 6a. On figure 6b the dimensionless parameters α and δ are expressed in terms of the local height $h(x)$ using relations (95). Figure 6a clearly shows the transition between the Regime A with $\delta = 1$ to the Regime B with $\delta = O\left(\varepsilon^2\right)$ through a Regime C (transition zone).

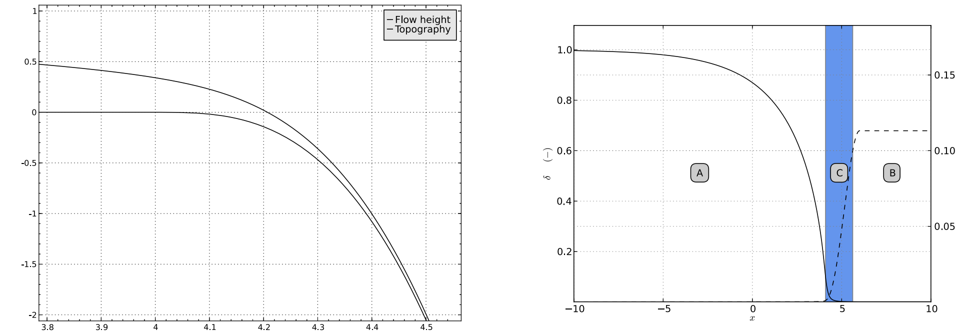
The corresponding hydraulic Froude number (defined by equation (69)) varies from $F_h \approx 1.7 \cdot 10^{-4}$ upstream to $F_h \approx 77$ downstream. The reduction of the wall friction transforms the sub-critical upstream flow ($F_h < 1$) to a super-critical flow ($F_h > 1$).

Case 1b.

Next we disturb the previous flow by adding at the bottom a varying topography i.e. $b(x) \neq 0$. The solution (see figure 7) shows that the steady state solution of Regime B is a translation of the bottom, whereas in Regime A there is a diffusion effect induced by the gravity contribution due to the local slope of the free-surface $\delta \partial_x H$.

12.3. Case 2: Multi-regime transition due to a change of the mean slope

We set the geometric scaling parameter ε=10^{-2}, the power-law exponent $p = 3$ and a constant friction coefficient at bottom $C = 50$. At the left half of the domain ($x < 4$), we set $\gamma = \varepsilon$ in order to obtain a Regime A ($\alpha = \varepsilon^2$ and $\delta = 1$). At the right half of the domain part ($6 < x$), we consider a different slope $\tan\theta_\infty = 75$. The slope varies smoothly from 0 to $\tan\theta_\infty$ in $4 \leq x \leq 6$ (see figure 8a). The resulting flow height is plotted on figure 8a, and the local dimensionless parameters are plotted on figure 8b.



(a) Zoom of the flow height $h(x)$ and the topography (x from 3.8 to 4.6)

(b) $\alpha(h)$ and $\delta(h)$ (full domain)

Figure 8: Case 2: Multi-regime transition due to a change of the mean slope

The unified model allows to shift from one regime to another continuously due to a change of the local slope from weak to large.

The corresponding "hydraulic" Froude number defined by equation (69), varies from $F_h \approx 0.1$ upstream to $F_h \approx 56$ downstream. The increase of the wall slope transforms the sub-critical upstream flow ($F_h < 1$) to a super-critical flow ($F_h > 1$).

These test cases present sharp variations of regime due to either a change of boundary condition at bottom, or a change of mean slope. In each case, the "unified model" (58) allows to compute the solution in all the domain, in other words the bi-regime solution. At contrary, to our knowledge no other single equation from the literature allows to handle such configurations.

13. Conclusion

In this paper we have derived one-equation models (lubrication type) and two-equation models (depth-integrated, shallow-water type) consistent with viscous multi-regime free surface flows, with varying basal boundary conditions. These models remain valid for the whole range of basal friction, from adherence to pure slip. A particular attention has been paid to geophysical flows e.g. ice or lava flows. At order 0, the derived models are of practicable use, while their complexity at order 1 may prevent to implement them for complex multi-scale geophysical flows. Nevertheless they remain interesting on regular topography, typically for reproductible experimental flows. The order 1 expressions of each field are required to derive consistent depth-integrated two-equation models, since it cannot be achieved using 0th order expressions only. The models and explicit expressions of all fields (velocity, pressure and stress components) are valid either in a mean slope coordinate system with local topography variations, or in the Cartesian coordinate system using the Prandtl shift, hence valid for any varying topography without the existence of a natural mean slope. The calculations have enhanced a natural limitation of the unified expressions and models: a vanishing friction case (pure slipe case) is possible with a vanishing mean-slope only. Formal error estimates on the asymptotical solutions (velocity, pressure, stress) have been stated. These error estimates demonstrate that if the asymptotic expansions are derived at order 1 or 2 then the global error of the fields remain 0th order with respect to the friction law at bottom.

Few versions of two-equation models have been derived. A first version not fully consistent (with the primitive model, Navier-Stokes power-law free-surface) since the basal shear stress is expanded at order 0 only (and not at 1st order), is proposed. This model allows to derive a quite simple gradually varied flow model. The latter allows to define a hydraulic-like Froude number and the corresponding sub-critical and supercritical flows. Numerical tests demonstrate the robustness of this new unified model, while no single shallow equation from the literature would allow to handle the configuration considered.

Next, a fully consistent two-equation model has been derived. The first natural version in conservative form turned out to be linearly ill-posed. Then, a quasi-linear form of this unified two-equation model (which could be called a three-equation model) is proposed since it is linearly well-posed. Hence this last model form is the right model of these newly derived multi-regime model families.
All these new multi-regime models include the classical ones from the literature if considering the same order, the same particular regime and/or the same condition at bottom (which are either no-slip or asymptotically vanishing friction).
The mathematical and numerical analyses of the present new models would deserve to be detailed; it will be done in forthcoming manuscripts.

The model derivation method was as follows. The asymptotic equations have been derived using a standard perturbation method of the Navier-Stokes equations free-surface, power-law, based on the algorithm formulated like in [19, 32]. The latter may allow to derive formally the asymptotic fields and equations up to order two, for all viscous regimes. Nevertheless, at order two, the expressions and model terms become extremely heavy in the general case hence likely not useful in practice for geophysical applications. (In the particular case, flat bottom with basal adherence, order two systems simplify greatly and may present great interests for fluid flow studies in lab experiments). In other respect, the well-known feature that in the case $p \geq 2$ (p, the power law exponent), the asymptotic expansions break down is retrieved (the pressure and normal stress components are not integrable); but this singularity is not formally penalizing in the present derivations since not appearing in the present 0th and 1st order model families. Furthermore, this unphysical singularity can be easily circumvent by regularizing the rheology law at vanishing shear rate (as it is classically done in the rheology literature).

Other rheology laws could be investigated such as yield stress fluids. For example, in [19], the authors derived a consistent shallow two-equation model for a Bingham fluid but at Regime C and with no-slip at bottom only. Following the present approach and the treatment of the yield stress presented in [19], multi-regime models based on multi basal boundary conditions at bottom could be obtained similarly for Bingham and Herschel-Buckley rheology laws. Also, the surface tension terms could be added easily to the present unified models (as it may be useful for industrial thin-film flows) using the method presented here (see e.g. [19] in a mono-regime case); it has not been done here since a particular attention has been paid to geophysical flows.
Finally, the present derivations and analysis can be extended to the 2D case; the basic derivation principles remain the same but the calculations are much heavier. Since many questions remain to be tackled in each model class presented here, 2D model analysis would need to focus on one aspect only (e.g. linear stability of the 2 and 3 equations models).

Acknowledgments. We would like to thank an anonymous reviewer for his detailed comments which have allowed to clarify few features of the new models.
The present work has partially been supported by the ADAGe project (ANR-09-SYSC-001) funded by the Agence National de la Recherche (ANR) (10 months postdoctoral fund of the first author who setted up and performed performed the numerical tests). Also, we would like to thank Mrs C. Ritz from CNRS - LGGE Grenoble who initiated the second author in 2007 into open questions related to the asymptotical models in ice-sheet modeling.

AppendixA. Asymptotic models for arbitrary topography: the Prandtl shift

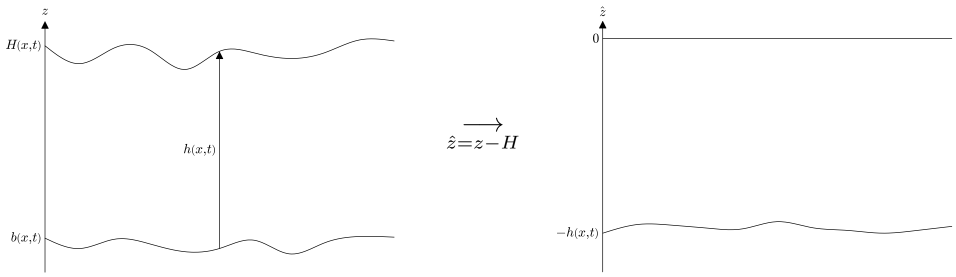
The equations have been written mainly in the mean slope coordinate system, hence a-priori limiting the equations validity to small local topography variation $\partial_x b$ (see the discussion led in Section 2; if local variations are large than the vertical velocity w would not remain small, and the scaling $W^* = \varepsilon U^*$ would not remain true anymore).

Nevertheless, the existence of a mean slope can be circumvented if using Prandlt's shift [34]. All the fields and equations, at the orders considered in the present study ie. order 1 in both regimes and order 1 in the geometrical ratio $\varepsilon = (\alpha\delta)^{1/2}$, are the same. Therefore, all the order 1 field expressions and equations stated in the present article are valid even if the existence of a meaningful mean slope is not clear.
In 3d, the extension of the mean slope concept would be a mean plan slope. Then similarly the extension of the Prandlt shift can be extended straightforwardly in presence of a transverse symmetry.

Let us recall the Prandtl transposition theorem [34] for the boundary layer equations. It states that, if the velocity satisfies the boundary layer equations, then applying a shift on the vertical component z with an arbitrary function $f(x,t)$ will also leave the boundary layer equations unchanged. Using this shift one can flatten the interface (see e.g. [35, 36]), and develop the terms related to the topography in the equations, see figure A.9. The Prandtl shift writes:

$$\hat{z} = z - H(x,t) \qquad \hat{w} = w - \partial_t H - u\partial_x H$$

The vertical shift transforms the free surface into a flat surface (see figure A.9). In the present study, its main interest is that all z-independent fields are identical in both coordinate systems. In other terms, the depth-averaged models do not depend on the Prandtl shift, and there is no need to make any reverse shift.



(a) (x,z) coordinate system

(b) Prandtl $(x,\hat{z})$ coordinate system

Figure A.9: The Prandtl shift flattens the free surface.

Let us introduce the basic rules for writing derivatives and integrals using the Prandtl shift. We denote the variables in the new coordinate system with a hat symbol: $\hat{\varphi}(x, \hat{z}(x,t), t) = \varphi(x,z,t)$. The derivatives with respect to x, z and t write in the new system:

$$\begin{aligned}
\partial_x \varphi(x,z,t) &= \partial_x \hat{\varphi}(x,\hat{z},t) - \partial_{\hat{z}} \hat{\varphi}(x,\hat{z},t)\,\partial_x H(x,t) \\
\partial_z \varphi(x,z,t) &= \partial_{\hat{z}} \hat{\varphi}(x,\hat{z},t) \\
\partial_t \varphi(x,z,t) &= \partial_t \hat{\varphi}(x,\hat{z},t) - \partial_{\hat{z}} \hat{\varphi}(x,\hat{z},t)\,\partial_t H(x,t)
\end{aligned}$$

Since the z-derivative remains unchanged in the new coordinate system, only the interval of the integrals changes:

$$\int_z^H \varphi \mathrm{d}z = \int_{\hat{z}}^0 \hat{\varphi} \mathrm{d}\hat{z} \qquad \int_b^z \varphi \mathrm{d}z = \int_{-h}^{\hat{z}} \hat{\varphi} \mathrm{d}\hat{z} \qquad \int_b^H \varphi \mathrm{d}z = \int_{-h}^0 \hat{\varphi} \mathrm{d}\hat{z}$$

But extra terms appear when deriving an integral:

$$\begin{aligned}
\partial_x \left(\int_z^H \varphi \mathrm{d}z \right)(x,z,t) &= \partial_x \left(\int_{\hat{z}}^0 \hat{\varphi} \mathrm{d}\hat{z} \right)(x,\hat{z},t) + \hat{\varphi}(x,\hat{z},t)\,\partial_x H(x,t) \\
\partial_t \left(\int_z^H \varphi \mathrm{d}z \right)(x,z,t) &= \partial_t \left(\int_{\hat{z}}^0 \hat{\varphi} \mathrm{d}\hat{z} \right)(x,\hat{z},t) + \hat{\varphi}(x,\hat{z},t)\,\partial_t H(x,t)
\end{aligned}$$

These relations are applied to the friction condition (24) $w_{|z=b} = u_{|z=b}\partial_x b$. The condition turns to be equivalent with the mass conservation equation (37) (which is invariant under the Prandtl shift):

$$\partial_t H + \partial_x \hat{q} = a \text{ with } \hat{q} = \int_{-h}^{0} \hat{u} \mathrm{d}\hat{z} \tag{A.1}$$

Likewise the functions ψ, see Eqn (33), become:

$$\begin{array}{rcl} \hat{\psi}_{xz}\left(\hat{\sigma}_{xx}, \hat{\pi}, \hat{u}, \hat{w}\right) & = & -\lambda\hat{z} - \delta\left(\hat{\pi}\partial_x H + \partial_x\left(\int_{\hat{z}}^0 \hat{\pi}\mathrm{d}\hat{z}\right)\right) \\ & & -\beta\left(\partial_t\left(\int_{\hat{z}}^0 \hat{u}\mathrm{d}\hat{z}\right) + \partial_x\left(\int_{\hat{z}}^0 \hat{u}^2\mathrm{d}\hat{z}\right) - \hat{u}\hat{w} - a\hat{u}_{|z=0}\right) \\ & & +\alpha\delta\left(\partial_x\left(\int_{\hat{z}}^0 \hat{\sigma}_{xx}\mathrm{d}\hat{z}\right) + \hat{\sigma}_{xx}\partial_x H\right) \end{array} \tag{A.2}$$

$$\begin{array}{rcl} \psi_{xx}\left(\hat{\sigma}_{xz}, \hat{\sigma}_{xx}, \hat{u}\right) & = & 2\left\|\hat{\sigma}\right\|^{1-p}\left(\partial_x \hat{u} - \partial_{\hat{z}}\hat{u}\partial_x H\right) \end{array}$$

$$\begin{array}{rcl} \hat{\psi}_{\pi}\left(\hat{\sigma}_{xz}, \hat{\sigma}_{xx}, \hat{u}, \hat{w}\right) & = & -\hat{z} - \alpha\left(\hat{\sigma}_{xx} + \partial_x\left(\int_{\hat{z}}^0 \hat{\sigma}_{xz}\mathrm{d}\hat{z}\right) + \hat{\sigma}_{xz}\partial_x H\right) \\ & & +\alpha\beta\left(\partial_t\left(\int_{\hat{z}}^0 \hat{w}\mathrm{d}\hat{z}\right) + \partial_x\left(\int_{\hat{z}}^0 \hat{u}\hat{w}\mathrm{d}\hat{z}\right) - \hat{w}^2 - a\hat{w}_{|\hat{z}=0}\right) \\ & & +\alpha\beta\partial_x H\left(\partial_t\left(\int_{\hat{z}}^0 \hat{u}\mathrm{d}\hat{z}\right) + \partial_x\left(\int_{\hat{z}}^0 \hat{u}^2\mathrm{d}\hat{z}\right) - \hat{u}\hat{w} - a\hat{u}_{|\hat{z}=0}\right) \\ & & +\alpha\beta\left(\partial_x^2 H \int_{\hat{z}}^0 \hat{u}^2\mathrm{d}\hat{z} + 2\partial_{tx}H\int_{\hat{z}}^0 \hat{u}\mathrm{d}\hat{z} - \hat{z}\partial_t^2 H\right) \\ & & +\alpha\beta\left(\hat{u} - \hat{u}_{|\hat{z}=0}\right)\partial_x H\partial_t H \end{array} \tag{A.3}$$

$$\begin{array}{rcl} \hat{\psi}_{bu}\left(\hat{\sigma}_{xz}, \hat{\sigma}_{xx}\right) & = & C\dfrac{\left(\left|\hat{\sigma}_{xz}\left(1-\alpha\delta\partial_x b^2\right) - 2\alpha\delta\hat{\sigma}_{xx}\partial_x b\right|^{p-1}\right)_{|\hat{z}=-h}}{\left(1+\alpha\delta\partial_x b^2\right)^{p+\frac{1}{2}}}\left(\hat{\sigma}_{xz|\hat{z}=-h}\left(1-\alpha\delta\partial_x b^2\right)\right. \\ & & \left. -2\alpha\delta\hat{\sigma}_{xx|\hat{z}=-h}\partial_x b\right) \\ \hat{\psi}_{u}\left(\hat{\sigma}_{xz}, \hat{\sigma}_{xx}, \hat{u}, \hat{w}\right) & = & \hat{u}_{|\hat{z}=-h} + \int_{-h}^{\hat{z}}\left(\left\|\hat{\sigma}\right\|^{p-1}\hat{\sigma}_{xz}\right)\mathrm{d}\hat{z} \\ & & -\alpha\delta\left(\partial_x\left(\int_{-b}^{\hat{z}}\hat{w}\mathrm{d}\hat{z}\right) + \left(\hat{z}+b\right)\partial_{tx}H + \partial_x b\partial_t H + \partial_x^2 H\int_{-b}^{\hat{z}}\hat{u}\mathrm{d}\hat{z}\right) \\ & & -\alpha\delta\partial_x H\left(\partial_x\left(\int_{-b}^{\hat{z}}\hat{u}\mathrm{d}\hat{z}\right) - \hat{w} - 2\partial_t H - \hat{u}\partial_x H + \hat{u}_{|\hat{z}=-h}\partial_x b\right) \\ & & -\alpha\delta\left(\partial_t H + \hat{w}_{|\hat{z}=-h}\partial_x b\right) \\ \hat{\psi}_{w}\left(\hat{\sigma}_{xz}, \hat{\sigma}_{xx}, \hat{u}, \hat{w}\right) & = & -\int_{\hat{z}}^0 \partial_x \hat{u}\mathrm{d}\hat{z} - a \end{array} \tag{A.4}$$

It is important to note that all the functions $\hat{\psi}_{\square}$ are unchanged under the Prandtl shift at 0th order and at order 1 *if considering 1st order terms in the geometrical ratio* ε (i.e. skipping terms in $\alpha\delta$, $\alpha\delta = \varepsilon^2$). This last simplification is crucial; otherwise many new terms appear.
Moreover the fixed point like formulation (35) and the iterative schemes (38)-(39) still apply.

Remark 29. Extra terms appear in the pressure expression at order 1 in the case $\alpha = 0(1)$ (Regime B) but this expression is useless to derive the present asymptotical models.

AppendixB. Discharge expression: first order terms

We present here the first order terms of the discharge q. The expressions are given in the mean slope coordinate system. For a sake of clarity, the mass source term a has been supposed to a constant: terms in $\partial_x a = 0$ have been skipped.

The discharge order 1 can be written as follows:

$$
\begin{aligned}
q^{(1)} &= q^{(0)} + \beta A_1(h,\Lambda)\partial_x h + \beta A_2(h,\Lambda)\partial_x\Lambda + \beta A_3(h,\Lambda)\partial_t\Lambda \\
&\quad + \beta A_4(h,\Lambda)\partial_x b + \beta A_5(h,\Lambda)a + \mathcal{O}((\beta+\alpha+\delta)^2+\varepsilon^2)
\end{aligned}
\tag{B.1}
$$

with $q^{(0)}$ defined by (50) and:

$$
\begin{aligned}
A_1(h,\Lambda) &= \left(2\frac{p^2}{(2+p)(2p+3)}h^3 + 6\frac{p^2(p+1)}{(2+p)(2p+3)}Ch^2 + \frac{p^2(3p+2)}{2+p}C^2h + p^3C^3\right)|\Lambda|^{p-1}\Lambda^{2p}h^{3p} \\
A_2(h,\Lambda) &= \left(2\frac{p^2(3p^2+4p-3)h^4}{(2p+3)(3p+4)(2+p)^2} + 2\frac{p^2(3p^2+4p-3)h^3}{(2p+3)(2+p)^2}C + \frac{p^2(3p-2)h^2}{2+p}C^2 + p^2(p-1)C^3h\right)|\Lambda|^{p-2}\Lambda^{2p}h^{3p} \\
A_3(h,\Lambda) &= -\left(2\frac{h^2}{(2+p)(2p+3)} + 2\frac{h}{2+p}C + C^2\right)p^2\Lambda^{2(p-1)}h^{2p+1} \\
A_4(h,\Lambda) &= -\left(\frac{h^2}{2(p+1)^2}C + \frac{h}{1+p}C^2 + C^3\right)p\Lambda^{3p-1}h^{3p} \\
A_5(h,\Lambda) &= p^2\left(C^2 + C\frac{(2p+3)}{(p+1)(p+2)}h + \frac{(2p+3)}{2(p+1)^2(p+2)}h^2\right)h^{2p}\Lambda^{2p-1}
\end{aligned}
\tag{B.2}
$$

Let us notice that $\partial_x\Lambda = O(\delta) = \partial_t\Lambda$. Also, the topography variation $\partial_x b$ appears a-priori in all terms $A_\square^{\varphi}$ since the definition of Λ.

Recall that the newtonian case corresponds to the case $p = 1$, and no-slip at bottom corresponds to the case $C = 0$.

AppendixC. Basal friction corrective terms in the two-equation model

We present here the basal shear stress first order corrective terms; the expressions are given in the mean slope coordinate system. Like for the discharge expression, for a sake of clarity, the mass source term a has been supposed to a constant: terms in $\partial_x a = 0$ have been skipped.

The expression of $\tau^{(\beta)}$ writes:

$$
\tau^{(\beta)} = B_1(h,\Lambda)\partial_x h + B_2(h,\Lambda)\partial_x\Lambda + B_3(h,\Lambda)\partial_t\Lambda + B_4(h,\Lambda)\partial_x b + B_5(h,\Lambda)a
$$

with:

$$
\left\{
\begin{aligned}
B_1(h,\Lambda) &= -\frac{p(Cp+h)}{(C(p+2)+h)(p+2)(2p+3)}\Lambda^{2p}h^{2(p+1)} \\
B_2(h,\Lambda) &= -\frac{p(3p^2+p-4)C + hp(3p-2)}{(2p+3)(3p+4)(2+p)(h+(2+p)C)}|\Lambda|\Lambda^{2p-2}h^{2p+3} \\
B_3(h,\Lambda) &= \frac{p}{(2p+3)(2+p)(h+(2+p)C)}|\Lambda|^{p-1}h^{p+3} \\
B_4(h,\Lambda) &= C\frac{(2+p)h + 2C(p+1)}{2(1+p)^2(h+(2+p)C)}|\Lambda|^p h^p \\
B_5(h,\Lambda) &= \frac{ph}{2(p+1)^2(h+(2+p)C)}|\Lambda|^p h^{p+1}
\end{aligned}
\right.
$$

with Λ defined by (49) hence $\partial_x\Lambda$ and $\partial_t\Lambda$ satisfying (54). Recall that $\partial_x\Lambda = O(\delta) = \partial_t\Lambda$.

The topography variation $\partial_x b$ appears a-priori in all terms $B_\square$ since the definition of Λ.

Recall the newtonian case corresponds to the case $p = 1$, and no-slip at bottom corresponds to the case $C = 0$.

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